Chapter 2 Infinite Series Generating Functions & Basic Hypergeometric Series

Let $p(m,n) = \#$ of partitions of n into m parts.

~~Let $p_k(m,n) = \#$ of partitions of n into m parts $\le k$.~~

Note If $m > n$ then $p(m,n) = 0$.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p(m,n)\, z^m q^n = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} p(m,n)\, z^m\right) q^n$$

$$= 1 + (z^1 q^1) + (z^1 q^2 + z^2 q^{1+1})$$

$$+ (z q^3 + z^2 q^{2+1} + z^3 q^{1+1+1}) + \cdots$$

$$= \sum_{\lambda \in P} z^{\#(\lambda)} q^{|\lambda|}$$

$$= 1 + (zq) + (z+z^2)q^2 + (z+z^2+z^3)q^3 + \cdots$$

$\vdots$

Let $p_k(m,n) = \#$ of ptns of n into m parts $\le k$

Each ptn of n into m parts $\le k$ can be be uniquely written

$\lambda : n = 1a_1 + 2a_2 + \cdots + k a_k$ where $a_i \geq 0$

where $a_1 + a_2 + \cdots + a_k = m$

Let P_k = set of ptns into parts $\le k$

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Then
$$\sum_{\lambda \in P_k} z^{\#(\lambda)} q^{|\lambda|}$$
$$= \sum_{a_1, a_2, \ldots, a_k \geq 0} z^{a_1+a_2+\cdots+a_k} q^{a_1+2a_2+\cdots+ka_k}$$
$$= \left(\sum_{a_1\geq 0} z^{a_1} q^{a_1}\right)\left(\sum_{a_2\geq 0} z^{a_2} q^{2a_2}\right)\cdots\left(\sum_{a_k\geq 0} z^{a_k} q^{ka_k}\right)$$
$$= (1 + zq^1 + z^2 q^{1+1} + \cdots)$$
$$(1 + zq^2 + z^2 q^{2+2} + \cdots)$$
$$\vdots$$
$$(1 + zq^k + z^2 q^{k+k} + \cdots)$$
$$= \left(\frac{1}{1-zq}\right)\left(\frac{1}{1-zq^2}\right)\cdots\left(\frac{1}{1-zq^k}\right)$$
provided $|zq| < 1$

Letting $k \to \infty$.
$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p(m,n) z^m q^n = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} p(m,n) z^m\right) q^n$$
$$= \prod_{k=1}^{\infty} \frac{1}{1-zq^k}$$
if $|q| < 1$ & $|zq| < 1$ ie ∎.

(3)

Let $H \subset \mathbb{N} = \{1, 2, \ldots\}$.

Let $p(``H", m, n) = \#$ of partition of n with m parts each from H.

Let $p(``H"(\leq d), m, n) = \#$ of partitions of n with m parts each part from H & each part occurs at most d times.

<u>Theorem</u> Let $|q|<1$ & $|zq|<1$. Then

(i) $$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p(``H", m, n) z^m q^n = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} p(``H", m, n) z^m\right) q^n$$
$$= \prod_{n\in H}(1 - z q^n)^{-1}.$$

(ii) $$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p(``H"(\leq d), m, n) z^m q^n$$
$$= \prod_{n\in H}(1 + z q^n + z^2 q^{2n} + \cdots + z^d q^{nd})$$
$$= \prod_{n\in H}\frac{(1 - z^{d+1} q^{(d+1)n})}{(1 - z q^n)}.$$

<u>Notation</u>: Let $q, a \in \mathbb{C}$, $|q|<1$. Let $n \geq 1$ $(n \in \mathbb{Z})$.

$$(a)_n := (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}) = \prod_{k=0}^{n-1}(1-aq^k),$$
$$(a)_0 := 1.$$
$$(a)_\infty = (a;q)_\infty = \lim_{n\to\infty}(a;q)_n = \prod_{k=0}^{\infty}(1-aq^k).$$

Hence
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p(m,n) z^m q^n = \prod_{n=1}^{\infty}\frac{1}{1-zq^n} = \frac{1}{(zq;q)_\infty} \quad \text{for } |q|<1, \ |zq|<1.$$

(4)

(Cauchy) *Theorem* (q-binomial theorem). If $|q|<1$ and $|z|<1$ then

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n = \frac{(az)_\infty}{(z)_\infty} = \frac{(1-az)(1-azq)\cdots}{(1-z)(1-zq)\cdots}$$

Proof Let $a, q \in \mathbb{C}$, $|q|<1$.

$$\frac{(az)_\infty}{(z)_\infty} = \prod_{n=0}^{\infty} \frac{(1-azq^n)}{(1-zq^n)} \quad \text{converges uniformly}$$

on compact subsets of $|z|<1$ and defines an analytic function for $|z|<1$.

ie $F(z) := \dfrac{(az)_\infty}{(z)_\infty} = \sum_{n=0} A_n z^n$ for $|z|<1$.

$$F(zq) = \prod_{n=0}^{\infty} \frac{(1-azq^{n+1})}{(1-zq^{n+1})} = \frac{(1-azq)(1-azq^2)\cdots}{(1-zq)(1-zq^2)\cdots}$$

So

$$\frac{(1-az)}{(1-z)} F(zq) = F(z), \ \&$$

$$(1-az)\, F(zq) = (1-z)\, F(z).$$

$$(1-z)\, F(z) = (1-z) \sum_{n=0}^{\infty} A_n z^n$$

$$= \sum_{n=0}^{\infty} A_n z^n - \sum_{n=0}^{\infty} A_n z^{n+1}$$

$$= \sum_{n=0}^{\infty} A_n z^n - \sum_{n=1}^{\infty} A_{n-1} z^n$$

$$= A_0 + \sum_{n=1}^{\infty} (A_n - A_{n-1}) z^n$$

(5)

$$(1-az)F(zq) = (1-az)\sum_{n=0}^{\infty} A_n (qz)^n$$

$$= \sum_{n=0}^{\infty} q^n A_n z^n - a\sum_{n=0}^{\infty} q^n z^{n+1} A_n$$

$$= A_0 + \sum_{n=1}^{\infty} q^n A_n z^n - a\sum_{n=1}^{\infty} q^{n-1} z^n A_{n-1}$$

$$= A_0 + \sum_{n=1}^{\infty} (q^n A_n - a q^{n-1} A_{n-1}) z^n,$$

Hence, for $n \geq 1$,

$$A_n - A_{n-1} = q^n A_n - a q^{n-1} A_{n-1}$$

$$(1-q^n) A_n = (1 - aq^{n-1}) A_{n-1}$$

$$A_n = \frac{(1-aq^{n-1})}{(1-q^n)} A_{n-1}$$

$$= \frac{(1-aq^{n-1})}{(1-q^n)} \frac{(1-aq^{n-2})}{(1-q^{n-1})} \frac{(1-aq^{n-3})}{(1-q^{n-2})} \cdots \frac{(1-a)}{(1-q)} A_0$$

$$F(0) = \frac{(0)_\infty}{(0)_\infty} = \frac{1}{1} = A_0.$$

Hence $A_n = \dfrac{(a;q)_n}{(q)_n} = \dfrac{(a)_n}{(q)_n}$, for $n \geq 1$, &

$$F(z) = \frac{(az)}{(z)_\infty} = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n. \quad \square$$

<u>Why q-binomial Theorem</u>?

Let $a = q^\alpha$ where $\alpha \in \mathbb{Z}$, $\alpha > 0$.

$$\frac{(a)_n}{(q)_n} = \frac{(1-q^\alpha)(1-q^{\alpha+1})\cdots(1-q^{\alpha+n-1})}{(1-q)(1-q^2)\cdots(1-q^n)}$$

(C)

$$= \frac{(1-q^{\alpha})(1-q^{\alpha+1})\cdots(1-q^{\alpha+n-1})}{(1-q)\,(1-q)\cdots(1-q)}$$

$$\cdot\ \frac{1}{1\cdot(1+q)(1+q+q^2)\cdots(1+q+\cdots q^{n-1})}$$

$$\lim_{q\to 1}\frac{1-q^j}{1-q} = \lim_{q\to 1}\frac{-jq^{j-1}}{-1} \qquad (\text{by } L'H)$$

$$= j \qquad (\text{if } j\in\mathbb{Z}).$$

Hence $\displaystyle\lim_{q\to 1^-}\frac{(a)_n}{(q)_n} = \frac{(\alpha)(\alpha+1)\cdots(\alpha+n-1)}{(1)\,(2)\cdots(n)}$

$$\frac{(az)_\infty}{(z)_\infty} = \frac{(1-q^{\alpha}z)(1-q^{\alpha+1}z)\cdots\cdots}{(1-z)(1-zq)\cdots(1-zq^{\alpha})\cdots}$$

$$= \frac{1}{(1-z)(1-zq)\cdots(1-zq^{\alpha-1})}$$

$$\lim_{q\to 1^-}\frac{(q^{\alpha}z)_\infty}{(z)_\infty} = \frac{1}{(1-z)^{\alpha}}.$$

Hence (formally), we have

$$1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}z^n = (1-z)^{-\alpha} \quad \text{for } |z|<1.$$

(Gen. Binomial Thm.).

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<u>Corollary (Euler)</u> Suppose $|q|<1$.

(1) $$\sum_{n=0}^{\infty} \frac{z^n}{(q)_n} = \frac{1}{(z)_\infty} \quad \text{for } |z|<1.$$

(2) $$\sum_{n=0}^{\infty} \frac{(-z)^n q^{n(n-1)/2}}{(q)_n} = (z)_\infty \quad \text{for all } z.$$

<u>Proof:</u>

(1) Let $a=0$ in q-bin. Thm gives
$$\sum_{n=0}^{\infty} \frac{z^n}{(q)_n} = \frac{1}{(z)_\infty}.$$

(2) In q-bin. Thm replace a by a/b & z by bz
$$\sum_{n=0}^{\infty} \frac{\left(\frac{a}{b}\right)_n}{(q)_n} \frac{(b^n z^n)}{1} = \frac{(az)_\infty}{(bz)_\infty} \quad \text{provided } |bz|<$$

$$\left(\frac{a}{b}\right)_n b^n = b^n\left(1-\frac{a}{b}\right)\left(1-\frac{a}{b}q\right)\left(1-\frac{a}{b}q^2\right)\cdots\left(1-\frac{a}{b}q^{n-1}\right)$$
$$= (b-a)(b-aq)(b-aq^2)\cdots(b-aq^{n-1})$$

$$\lim_{b\to 0} \left(\frac{a}{b}\right)_n b^n = (-a)(-aq)\cdots(-aq^{n-1})$$
$$= (-1)^n a^n q^{n(n-1)/2}.$$

It can be shown that the series (let z, q, a be fixed, $|q|<1$)
$$\sum_{n=0}^{\infty} \left(\frac{a}{b}\right)_n b^n z^n \quad \text{converges uniformly}$$
in b (for b on some disk $|b|\le\delta$). Hence
$$\lim_{b\to 0} \sum_{n=0}^{\infty} \left(\frac{a}{b}\right)_n b^n z^n = \sum_{n=0}^{\infty} \lim_{b\to 0} \left(\frac{a}{b}\right)_n b^n z^n$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{n(n-1)/2} z^n}{(q)_n}$$

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Similarly
$$\lim_{b\to 0} \frac{(az)_\infty}{(bz)_\infty} = \frac{(az)_\infty}{(0)_\infty} = (az)_\infty.$$

Hence
$$\sum_{n=0}^{\infty} \frac{(-az)^n\, q^{n(n-1)/2}}{(q)_n} = (az)_\infty .$$

Replacing az by z gives the result. $\square$

<u>Combinatorial Proof of (1)</u> Replacing z by zq we write (1) in the equivalent form

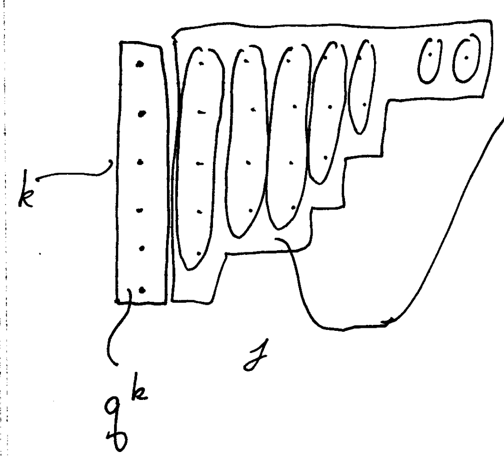
$$\sum_{k=0}^{\infty} \frac{q^k z^k}{(q)_k} = \frac{1}{(zq)_\infty} = \prod_{k=1}^{\infty} \frac{1}{1-zq^k},$$

(provided $|q|<1$ & $|zq|<1$).

Recall $\displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p(m,n)\, z^m q^n = \prod_{k=1}^{\infty} \frac{1}{1-zq^k}$

$$\sum_{\lambda\in\mathcal{P}} z^{\#(\lambda)} q^{|\lambda|}$$

Let $k\geq 1$. Let $\tilde{P}_k$ be the set of partitions into k parts.



reading columns to the right of the first col. we obtain partition with parts $\leq k$

& $GF = \dfrac{1}{(1-q)}\dfrac{1}{(1-q^2)}\cdots\dfrac{1}{(1-q^k)}$

$= \dfrac{1}{(q)_k}$

(9)

Hence
$$\sum_{\lambda\in \tilde{P}_k} z^{\#(\lambda)} q^{|\lambda|} = \frac{z^k q^k}{(q)_k}$$
since
$$P = \cup \tilde{P}_k \quad (\text{disjoint})$$
$$\sum_{\lambda\in P} z^{\#(\lambda)} q^{|\lambda|} = 1 + \sum_{k=1}^{\infty} \sum_{\lambda\in P_k} z^{\#(\lambda)} q^{|\lambda|}$$
Hence
$$\frac{1}{(zq)_\infty} = 1 + \sum_{k=1}^{\infty} \frac{z^k q^k}{(q)_k}.$$

<u>Combinatorial Proof of (2)</u> Replacing z by $-zq$ we write (2) in the equivalent form
$$1 + \sum_{k=1}^{\infty} \frac{z^k q^{k(k+1)/2}}{(q)_k} = (-zq)_\infty$$
$$= \prod_{m=1}^{\infty} (1 + z q^m)$$

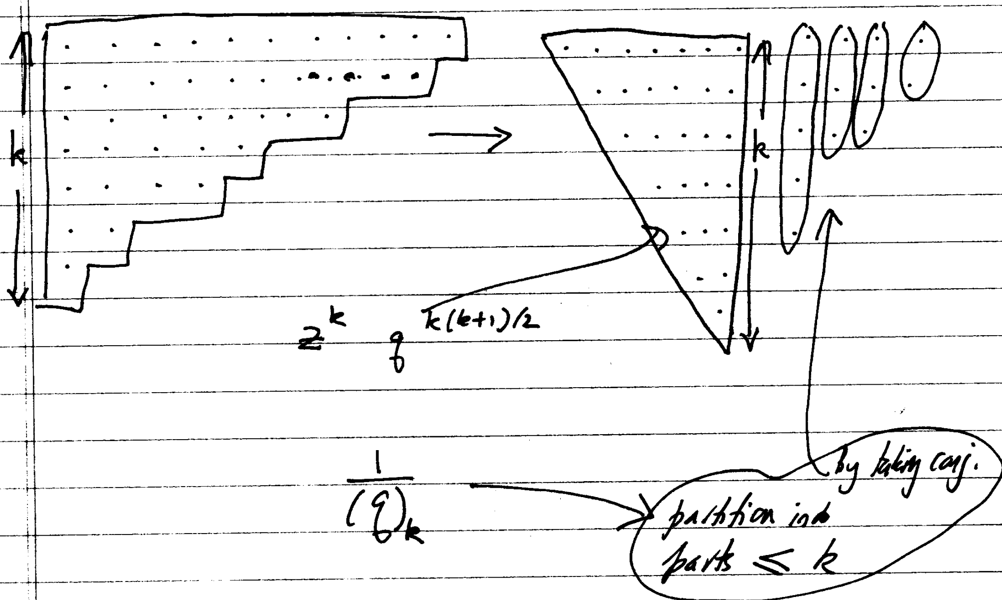
Generating function of ~~set of~~ partitions into distinct parts

Let $k\geq 1$. Let PD_k = set of partitions into exactly k distinct parts.

(Eg: $k=7$
$\lambda = (11, 10, 8, ~~6~~ 5, 4, 2, 1)$ of n
We remove 1 from smallest part
2 from next part

(10)

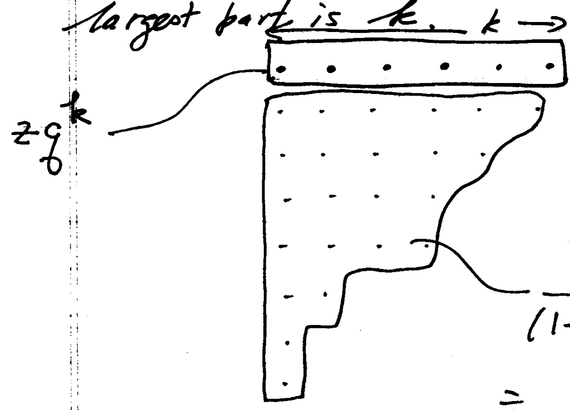
3 from next, ..., 7 from largest part
to get
$$\tilde{\lambda} = (4,4,3,1,1,0,0)$$
we obtain a partition of $n-(1+2+\cdots+7)$ into
$m \leq 7$ parts.)



Since
$$PD = \cup PD_k \qquad (\text{disjoint})$$
we have
$$(-zq)_\infty = \prod_{m=1}^{\infty}(1+zq^m) = 1 + \sum_{\lambda \in PD_k} z^{\#(\lambda)} q^{|\lambda|}$$
$$= 1 + \sum_{k=1}^{\infty} \frac{z^k q^{k(k+1)/2}}{(q)_k} . \ \square$$

Another identity for $1/(zq)_\infty$ (10A)

Let $k \geq 1$. Let $\tilde{P}'_k$ be the set of partitions whose largest part is k.



If we remove the largest part, what remains is a partition into parts $\leq k$

$$\frac{1}{(1-zq)}\,\frac{1}{(1-zq^2)}\cdots\frac{1}{(1-zq^k)} = \frac{1}{(zq)_k}$$

Hence

$$\sum_{\lambda \in \tilde{P}'_k} z^{\#(\lambda)} q^{|\lambda|} = \frac{zq^k}{(zq)_k}.$$

Since

$$P = \cup\, \tilde{P}'_k \qquad \text{(disjoint)}$$

$$\sum_{\lambda \in P} z^{\#(\lambda)} q^{|\lambda|} = \sum_{k \geq 0} \sum_{\lambda \in P_k} z^{\#(\lambda)} q^{|\lambda|}$$

&

$$\boxed{1 + \sum_{k=1}^{\infty} \frac{z q^k}{(zq)_k} = \frac{1}{(zq)_\infty}}$$

for $|q|<1$ (& $|zq|<1$)

Heine's Transformation (1847) (11)

Suppose $|q|<1$, $|t|<1$ & $|b|<1$, $b\neq 0$.

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n t^n}{(c)_n (q)_n} = \frac{(b)_\infty (at)_\infty}{(c)_\infty (t)_\infty} \sum_{n=0}^{\infty} \frac{(\frac{c}{b})_n (t)_n b^n}{(at)_n (q)_n}$$

Proof

$$(cq^n)_\infty = (1-cq^n)(1-cq^{n+1})(1-cq^{n+2})\cdots$$

$$(c)_n = (1-c)(1-cq)\cdots(1-cq^{n-1}).$$

Hence

$$(c)_n (cq^n)_\infty = (c)_\infty \quad \&$$

$$(c)_n = \frac{(c)_\infty}{(cq^n)_\infty}, \quad (cq^n)_\infty = \frac{(c)_\infty}{(c)_n}$$

Hence

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n t^n}{(c)_n (q)_n} = \sum_{n=0}^{\infty} (a)_n \frac{(b)_\infty}{(bq^n)_\infty} \cdot \frac{(cq^n)_\infty}{(c)_\infty} \cdot \frac{1}{(q)_n} t^n$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n (cq^n)_\infty}{(bq^n)_\infty (q)_n} t^n$$

$$\boxed{\frac{(at)_\infty}{(t)_\infty} = \sum_{m=0}^{\infty} \frac{(a)_m t^m}{(q)_m}}$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n t^n}{(q)_n} \frac{(cq^n)_\infty}{(bq^n)_\infty}$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n t^n}{(q)_n} \sum_{m=0}^{\infty} \frac{(c/b)_m (bq^n)^m}{(q)_m}$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a)_n t^n (c/b)_m}{(q)_n (q)_m} b^m q^{mn}$$

(12)

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(c/b)_m\, b^m}{(q)_m} \frac{(a)_n (q^m t)^n}{(q)_n}$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{m=0}^{\infty} \frac{(c/b)_m\, b^m}{(q)_m} \sum_{n=0}^{\infty} \frac{(a)_n\, (q^m t)^n}{(q)_n}$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{m=0}^{\infty} \frac{(c/b)_m\, b^m}{(q)_m} \frac{(aq^m t)_\infty}{(tq^m)_\infty}$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{m=0}^{\infty} \frac{(c/b)_m\, b^m}{(q)_m} \frac{(at)_\infty}{(at)_m} \cdot \frac{(t)_m}{(t)_\infty}$$

$$= \frac{(b)_\infty (at)_\infty}{(c)_\infty (t)_\infty} \sum_{m=0}^{\infty} \frac{(c/b)_m (t)_m\, b^m}{(at)_m (q)_m}. \qquad \Box$$

<u>Corollary</u> (Heine) Suppose $|q| < 1$ & $|c| < |ab|$.
Then

$$1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (q)_n} \left(\frac{c}{ab}\right)^n = \frac{(c/a)_\infty (c/b)_\infty}{(c)_\infty (c/ab)_\infty}$$

<u>Proof:</u> By Heine's transformation,

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (q)_n} \left(\frac{c}{ab}\right)^n = \frac{(b)_\infty \left(\frac{c}{b}\right)_\infty}{(c)_\infty \left(\frac{c}{ab}\right)_\infty} \sum_{n=0}^{\infty} \frac{\cancel{\left(\frac{c}{b}\right)_n} \left(\frac{c}{ab}\right)_n b^n}{\cancel{\left(\frac{c}{b}\right)_n} (q)_n}$$

(if $|q|<1$, $|c|<|ab|$ & $|b|<1$)

$$= \frac{(b)_\infty \left(\frac{c}{b}\right)_\infty}{(c)_\infty \left(\frac{c}{ab}\right)_\infty} \frac{\left(\frac{c}{a}\right)_\infty}{(b)_\infty} \qquad \text{(by } q\text{-bin. Thm)}$$

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$$= \frac{\left(\frac{c}{a}\right)_\infty \left(\frac{c}{b}\right)_\infty}{(c)_\infty \left(\frac{c}{ab}\right)_\infty}.$$

Result holds for general b ($b \neq 0$) by analytic continuation. $\square$

Corollary (Bailey) If $|q| < \min(1, |b|)$ then

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{\left(\frac{aq}{b}\right)_n (q)_n} \left(\frac{-q}{b}\right)^n = \frac{(aq;q^2)_\infty (-q;q)_\infty \left(\frac{aq^2}{b^2};q^2\right)_\infty}{\left(\frac{aq}{b};q\right)_\infty \left(-\frac{q}{b};q\right)_\infty}$$

Proof:

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n \left(\frac{-q}{b}\right)^n}{\left(\frac{aq}{b}\right)_n (q)_n} = \sum_{n=0}^{\infty} \frac{(b)_n (a)_n \left(\frac{-q}{b}\right)^n}{\left(\frac{aq}{b}\right)_n (q)_n}$$

$$= \frac{(a)_\infty (-q)_\infty}{\left(\frac{aq}{b}\right)_\infty \left(\frac{-q}{b}\right)_\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{q}{b}\right)_n \left(\frac{-q}{b}\right)_n}{(-q)_n (q)_n} a^n$$

(if $|q| < 1$, $\left|\frac{q}{b}\right| < 1$ & $|a| < 1$)

$$= \frac{(a)_\infty (-q)_\infty}{\left(\frac{aq}{b}\right)_\infty \left(\frac{-q}{b}\right)_\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{q^2}{b^2}; q^2\right)_n}{(q^2;q^2)_n} a^n$$

$$= \frac{(a)_\infty (-q)_\infty}{\left(\frac{aq}{b}\right)_\infty \left(\frac{-q}{b}\right)_\infty} \frac{\left(\frac{aq^2}{b^2}; q^2\right)_\infty}{(a;q^2)_\infty}$$

(14)

$$= \frac{(q;q^2)_\infty\,(aq;q^2)_\infty\,(-q)_\infty\,\left(\frac{aq^2}{b^2};q^2\right)_\infty}{\left(\frac{aq}{b}\right)_\infty\,(-q/b)_\infty\,(q;q^2)_\infty}$$

$$= \frac{(aq;q^2)_\infty\,(-q;q)_\infty\,(aq^2/b^2;q^2)_\infty}{\left(\frac{aq}{b};q\right)_\infty\,\left(\frac{-q}{b}\right)_\infty}.$$

Result holds for $|q| < \min(1, |b|)$ & general a (by analytic continuation on a). $\square$

<u>Corollary</u> Suppose $|q|<1$.

(1) $$\sum_{n=0}^{\infty} \frac{z^n q^{n^2-n}}{(q)_n (z)_n} = \frac{1}{(z)_\infty} \qquad \text{(Cauchy)}$$

(2) $$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n^{\,2}} = \frac{1}{(q)_\infty} \qquad \text{(Euler)}$$

<u>Proof</u>: In Heine's Corollary, let $a=\alpha^{-1}$, $b=\beta^{-1}$, $c=z$.

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{\alpha}\right)_n \left(\frac{1}{\beta}\right)_n \alpha^n \beta^n z^n}{(z)_n\,(q)_n} = \frac{(z\alpha)_\infty\,(z\beta)_\infty}{(z)_\infty\,(z\alpha\beta)_\infty}$$

provided $|q|<1$ & $|\alpha\beta z|<1$.

$$\cancel{\lim_{\alpha,\beta\to 0} \left(\tfrac{1}{\alpha}\right)_n \left(\tfrac{1}{\beta}\right)_n \alpha^n\beta^n = \lim_{\alpha,\beta\to 0} (\alpha-1)\ldots}$$

$$\left(\frac{1}{\alpha}\right)_n \alpha^n = (1-\tfrac{1}{\alpha})(1-q/\alpha)\cdots(1-q^{n-1}/\alpha)\alpha^n \quad (15)$$

$$= (\alpha-1)(\alpha-q)\cdots\cdots(\alpha-q^{n-1})$$

So $\lim_{\alpha\to 0} \left(\frac{1}{\alpha}\right)_n \alpha^n = (-1)^n q^{n(n-1)/2}$.

It follows that

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{\beta}\right)_n \beta^n z^n q^{n(n-1)/2}}{(z)_n (q)_n} = \frac{(z\beta)_\infty}{(z)_\infty}.$$

If we let $\beta \to 0$ we find that

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} (-1)^n q^{n(n-1)/2} z^n}{(z)_n (q)_n} = \frac{1}{(z)_\infty}$$

$$\& \quad \sum_{n=0}^{\infty} \frac{z^n q^{n^2-n}}{(z)_n (q)_n} = \frac{1}{(z)_\infty}.$$

(2) follows from (1) by letting $z=q$. $\square$

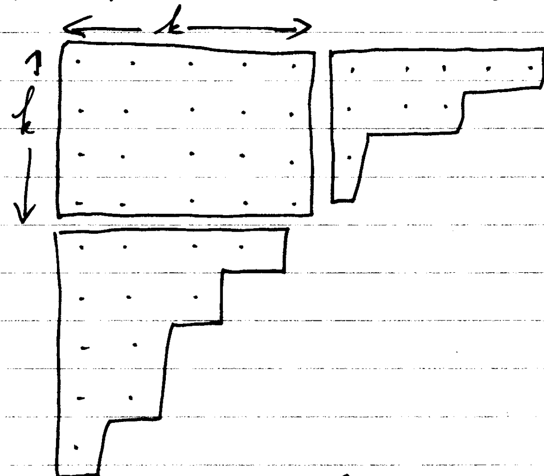
COMBINATORIAL PROOF of (1). Replacing z by zq we prove the equivalent statement that

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q)_n (zq)_n} = \frac{1}{(zq)_\infty}.$$

Recall that for $|z|<1$, $|zq|<1$ we have

$$\sum_{\lambda\in\mathcal{P}} z^{\#(\lambda)} q^{|\lambda|} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p(m,n) z^m q^n = \frac{1}{(zq)_\infty}.$$

For a partition λ ~~we let k be the largest square~~
Let k be the side of the largest square inside the top left part of the Ferrers diagram of λ.



We say λ has a Durfee square of side k.
Let k be fixed. Let $\mathcal{P}_{(k)}$ be set of partitions with Durfee square of side k. The right of the Durfee square we have a ptn with parts $\leq k$ by reading columns. Below the Durfee square we have a partition with parts $\leq k$.

$$\sum_{\lambda \in \mathcal{P}_{(k)}} z^{\#(\lambda)} q^{|\lambda|} = z^k q^{k^2} \times \frac{1}{(q)_k} \times \frac{1}{(zq)_k}$$

Since $\mathcal{P} = \cup\, \mathcal{P}_{(k)}$ (disjoint).
We have

$$\frac{1}{(zq)_\infty} = \sum_{\lambda \in \mathcal{P}} z^{\#(\lambda)} q^{|\lambda|} = \sum_{k=0}^{\infty} \sum_{\lambda \in \mathcal{P}(k)} z^{\#(\lambda)} q^{|\lambda|}$$

$$= \sum_{k=0}^{\infty} \frac{z^k q^{k^2}}{(zq)_k (q)_k} . \square$$

(17)

<u>Corollary</u> If $|q|<1$, then

$$\sum_{n=0}^{\infty} \frac{(a)_n q^{n(n+1)/2}}{(q)_n} = (aq;q^2)_\infty(-q;q)_\infty$$

<u>Proof:</u> Suppose $|q|<1$. In Corollary (Bailey), let $b=\beta^{-1}$

$$\sum_{n=0}^{\infty} \frac{(a)_n (\frac{1}{\beta})_n (-q\beta)^n}{(q)_n (a\beta q)_n} = \frac{(aq;q^2)_\infty(-q)_\infty(a\beta^2q^2;q^2)_\infty}{(a\beta q)_\infty(-\beta q)_\infty}$$

$$\lim_{\beta\to 0} (\tfrac{1}{\beta})_n(-\beta)^n = \lim_{\beta\to 0} (\beta-1)(\beta-q)\cdots(\beta-q^{n-1})(-1)^n$$

$$= (-1)^n q^{n(n-1)/2}(-1)^n = q^{n(n-1)/2}$$

So $$\sum_{n=0}^{\infty} \frac{(a)_n q^{n(n+1)/2}}{(q)_n} = (aq;q^2)_\infty(-q)_\infty . \quad \square$$

<u>JACOBI'S TRIPLE PRODUCT IDENTITY</u>

If $z\neq 0$, $|q|<1$, then

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty(q^2;q^2)_\infty .$$

<u>Note:</u> $$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = 1+\sum_{n=1}^{\infty} z^n q^{n^2} + \sum_{n=-1}^{-\infty} z^n q^{n^2}$$

$$= 1+\sum_{n=1}^{\infty} z^n q^{n^2} + \sum_{n=1}^{\infty} z^{-n} q^{(-n)^2}$$

$$= 1+\sum_{n=1}^{\infty} (z^n+z^{-n}) q^{n^2}.$$

(18)

Proof: Suppose $|q|<1$ & $|q|<|z|$.
We have

$$\sum_{n=0}^{\infty} \frac{z^n q^{\frac{1}{2}n(n-1)}}{(q)_n} = (-zq)_\infty$$

(by Euler's Cor. to q-bin). So

$$\sum_{n=0}^{\infty} \frac{z^n q^{n(n+1)}}{(q^2;q^2)_n} = (-zq^2;q^2)_\infty .$$

Replace z by zq^{-1} & we have

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n} = (-zq;q^2)_\infty .$$

So

$$(-zq;q^2)_\infty = \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n} \qquad (*)$$

$$= \sum_{n=0}^{\infty} \frac{z^n q^{n^2} (q^{2n+2};q^2)_\infty}{(q^2;q^2)_n (q^{2n+2};q^2)_\infty}$$

$$= \sum_{n=0}^{\infty} \frac{z^n q^{n^2} (q^{2n+2};q^2)_\infty}{(q^2;q^2)_\infty}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} z^n q^{n^2} (q^{2n+2};q^2)_\infty$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n^2} (q^{2n+2};q^2)_\infty$$

if $n<0$

(since $(q^{2n+2};q^2)_\infty = (1-q^{2n+2})\cdots(1-q^0)\cdots = 0$)

(19)

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n^2} \sum_{m=0}^{\infty} (-1)^m q^{m(2n+1)} \frac{q^{m^2}}{(q^2;q^2)_m}$$

(by (*) with $z = -q^{2n+1}$)

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^m z^n q^{m^2+2nm+m+n^2}}{(q^2;q^2)_m}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^m z^n q^{(m+n)^2+m}}{(q^2;q^2)_m}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^m z^{-m}}{(q^2;q^2)_m} \sum_{n=-\infty}^{\infty} z^{m+n} q^{(m+n)^2}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^m z^{-m}}{(q^2;q^2)_m} \sum_{k=-\infty}^{\infty} z^k q^{k^2}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{k=-\infty}^{\infty} z^k q^{k^2} \sum_{m=0}^{\infty} \frac{(-q/z)^m}{(q^2;q^2)_m}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n^2} \frac{1}{(-q/z;q^2)_\infty}$$ (by Euler's Cor. to q-bin.)

Therefore, $$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty$$

for $|\frac{q}{z}| < 1$. Result holds for ~~$z \neq 0$~~ by analytic continuation. $\square$

(29)

Suppose $0 < Q < 1$ & $Z \neq 0$.
Let $q = \sqrt{Q} = Q^{1/2}$, $z = ZQ^{1/2}$. By J.T.P. we have

$$\sum_{n=-\infty}^{\infty} Z^n Q^{n/2} Q^{n^2/2} = (-ZQ^{1/2}\cdot Q^{1/2}; Q)_\infty (-Z^{-1}Q^{-1/2}Q^{1/2}; Q)_\infty (Q;Q)_\infty$$

and

$$\sum_{n=-\infty}^{\infty} Z^n Q^{n(n+1)/2} = (-ZQ;Q)_\infty (-Z^{-1};Q)_\infty (Q;Q)_\infty .$$

By analytic continuation in q we have

$$(*) \quad \sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2} = (-zq;q)_\infty (-z^{-1};q)_\infty (q;q)_\infty$$

for $|q| < 1$.

<u>COR</u> (Euler's P.N.T).
For $|q|<1$,
$$\prod_{n=1}^{\infty} (1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}.$$

<u>Proof</u>: In $(*)$, replace q by q^3 and let $z = -q^{-1}$.
So
$$-zq \longrightarrow (-)(-q^{-1})q^3 = q^2$$
$$-z^{-1} \longrightarrow q \quad \&$$

$$\sum_{n=-\infty}^{\infty} (-q)^{-n} q^{\frac{1}{2}(3n^2+3n)} = (q^2;q^3)_\infty (q;q^3)_\infty (q^3;q^3)_\infty$$

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}(3n^2-n)} = \prod_{n=0}^{\infty} (1-q^{3n+2})(1-q^{3n+1})(1-q^{3n+3})$$

Replacing n by $-n$ in the sum we have
$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}(3n^2+n)} = \prod_{n=1}^{\infty} (1-q^n). \quad \square$$

COR: (Gauss) For $|q|<1$,

(1) $$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{m=1}^{\infty} \frac{(1-q^m)}{(1+q^m)} = \prod_{m=1}^{\infty} \frac{(1-q^m)^2}{(1-q^{2m})}.$$

(2) $$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \prod_{m=1}^{\infty} \frac{(1-q^{2m})}{(1-q^{2m-1})} = \prod_{m=1}^{\infty} \frac{(1-q^{2m})^2}{(1-q^m)}$$

Proof. In JTP let $z=-1$ we obtain

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = (q;q^2)_\infty (q;q^2)_\infty (q^2;q^2)_\infty$$

$$= (q;q^2)_\infty (q)_\infty$$

$$= \frac{(q)_\infty}{(q^2;q^2)_\infty} (q)_\infty$$

$$= \prod_{m=1}^{\infty} \frac{(1-q^m)^2}{(1-q^{2m})} = \prod_{m=1}^{\infty} \frac{(1-q^m)^2}{(1-q^m)(1+q^m)}$$

$$= \prod_{m=1}^{\infty} \frac{(1-q^m)}{(1+q^m)}.$$

Now $$\sum_{n=-\infty}^{\infty} q^{n(n+1)/2} = \sum_{n=0}^{\infty} q^{n(n+1)/2} + \sum_{n=-1}^{\infty} q^{n(n+1)/2}$$

$$= \sum_{n=0}^{\infty} q^{n(n+1)/2} + \sum_{m=0}^{\infty} q^{(-m-1)(-m)/2}$$ let $m=-n-1$ (ie $n=-m-1$)

(22)

$$= \sum_{n=0}^{\infty} q^{n(n+1)/2} + \sum_{m=0}^{\infty} q^{m(m+1)/2}.$$

Hence $\displaystyle\sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{1}{2}\sum_{n=-\infty}^{\infty} q^{n(n+1)/2}$

$$= \frac{1}{2}\,(-q;q)_\infty(-1;q)_\infty(q;q)_\infty \quad (\text{by } z=1 \text{ in } (*))$$

$$= \tfrac{1}{2}(-1;q)_\infty(-q;q)_\infty(q;q)_\infty$$

$$= \tfrac{1}{2}(2)(-q;q)_\infty(q^2;q^2)_\infty$$

$$= \prod_{m=1}^{\infty}(1+q^m)(1-q^{2m})$$

$$= \prod_{m=1}^{\infty}\frac{(1-q^{2m})}{(1-q^{2m-1})} = \prod_{m=1}^{\infty}\frac{(1+q^m)(1-q^{2m})(1-q^m)}{(1-q^m)}$$

$$= \prod_{m=1}^{\infty}\frac{(1-q^{2m})^2}{(1-q^m)}. \quad \square$$

<u>COR.</u> (Jacobi) If $|q|<1$, then

$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)\,q^{n(n+1)/2}.$$

<u>Proof</u>: By JTP(*),

$$\sum_{n=-\infty}^{\infty}(-1)^n z^n q^{n(n+1)/2} = (zq)_\infty(z^{-1})_\infty(q)_\infty$$

$$\sum_{n=-\infty}^{\infty}(-1)^n z^n q^{n(n+1)/2} = \sum_{n=0}^{\infty}(-1)^n z^n q^{n(n+1)/2} + \sum_{n=-\infty}^{-1}(-1)^n z^n q^{n(n+1)/2} \tag{23}$$

$(m=-n-1,\ n=-m-1)$

$$= \sum_{n=0}^{\infty}(-1)^n z^n q^{n(n+1)/2} + \sum_{m=0}^{\infty}(-1)^{-m-1} z^{-m-1} q^{(-m-1)(-m)/2}$$

$$= \sum_{n=0}^{\infty}(-1)^n z^n q^{n(n+1)/2} + \sum_{m=0}^{\infty}(-1)^{m+1} z^{-m-1} q^{m(m+1)/2}$$

$$= \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}(z^n - z^{-n-1})$$

$$= \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}(z^{-n-1})(z^{2n+1}-1), \quad \text{for } z \neq 0.$$

Since

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2} z^{-n-1}(z^{2n+1}-1) = (1-z^{-1})(z^{-1}q)_\infty(zq)_\infty(q)_\infty$$

and

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2} z^{-n-1}\frac{(z^{2n+1}-1)}{(1-z^{-1})} = (z^{-1}q)_\infty(zq)_\infty(q)_\infty$$

for $z \neq 0, 1$.

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2} z^{-n}\frac{(z^{2n+1}-1)}{(z-1)} = (z^{-1}q)_\infty(zq)_\infty(q)_\infty$$

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2} z^{-n}(1+z+z^2+\cdots+z^{2n}) = (z^{-1}q)(zq)_\infty(q)_\infty$$

Letting $z \to 1$ we find that

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} = (q)_\infty^3 = \prod_{n=1}^{\infty}(1-q^n)^3. \quad \square$$

(24)

COMBINATORIAL PROOF of JTP We write JTP in the equivalent form

$$(-zq;q)_\infty(-z^{-1};q)_\infty = \frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2}$$

or

$$(**)\qquad \prod_{n=1}^{\infty}(1+zq^n)(1+z^{-1}q^{n-1}) = \frac{1}{\prod_{m=1}^{\infty}(1-q^m)}\sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2}$$

The coeff of $z^k q^N$ in LHS

= # of pairs $(a_1,a_2,\dots,a_m)$, $(b_1,b_2,\dots,b_l)$

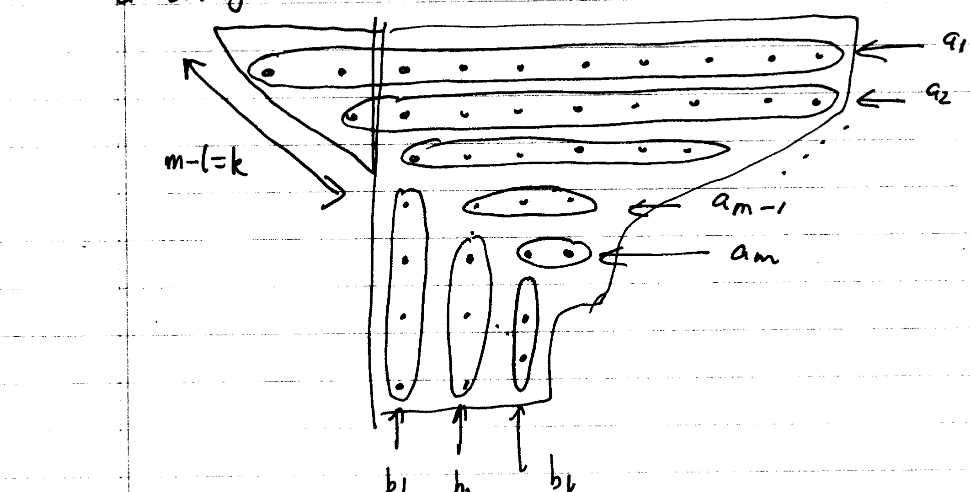
such that $a_1 > a_2 > \cdots > a_m \geq 1$

$b_1 > b_2 > \cdots > b_l \geq 0$

and $m - l = k$ and

$(a_1+a_2+\cdots+a_m) + (b_1+b_2+\cdots+b_l) = N$.

Suppose $k \geq 0$. For each such pair we form a diagram



This gives rise to a partition of $N - \frac{(k)(k+1)}{2}$. Hence (25)

$$\text{Coeff of } z^k q^N = \text{Coeff of } z^k q^N \text{ in } \frac{z^k q^{k(k+1)/2}}{\prod_{n=1}^{\infty}(1-q^n)}$$

Similarly result also holds for $k<0$, & we obtain (**). □

$\boxed{\text{A Generalization of Eulers Theorem that } p(O,n) = p(D,n)}$

<u>Theorem</u> (Sylvester)
Let $k \geq 1$.
Let $A_k(n)$ = the number of partitions of n into odd parts (repitition allowed) into exactly k different parts.
Let $B_k(n)$ = the number of partitions of n into exactly k noncontiguous sequences of one or more consecutive integers.
Then $A_k(n) = B_k(n)$ for all n.

<u>Example</u> $n=14$, $k=3$.
Partitions of $n=14$ into odd parts (repitition allowed) with 3 different parts:

$9+3+1+1$ So $A_3(14) = 7$
$7+5+1+1$
$7+3+3+1$
$7+3+1+1+1+1$
$5+5+3+1$
$5+3+3+1+1+1$
$5+3+1+1+1+1+1+1$

Partitions of $n=14$ into exactly k noncontiguous sequences of one or more consecutive integers: (26)

$10+3+1$

$9+4+1$

$8+4+2$

$8+5+1$

$7+5+2$

$7+4+\underline{2+1}$

$6+\underline{4+3}+1$

Suppose $|q|<1$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_k(n)\, a^k q^n$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} A_k(n)\, a^k\right) q^n$$

$$= \prod_{j=1}^{\infty}\left(1 + aq^{2j-1} + aq^{2(2j-1)} + aq^{3(2j-1)} + \cdots\right)$$

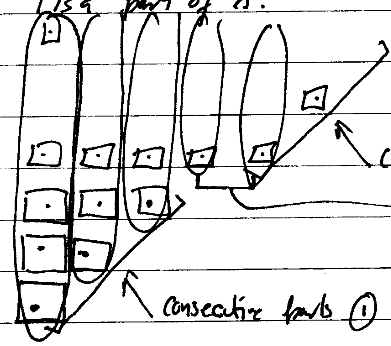
$$= \prod_{j=1}^{\infty}\left(1 + \frac{aq^{2j-1}}{1-q^{2j-1}}\right)$$

$$= \prod_{j=1}^{\infty}\frac{\left(1-q^{2j-1}+aq^{2j-1}\right)}{\left(1-q^{2j-1}\right)} = \prod_{j=1}^{\infty}\frac{1-(1-a)q^{2j-1}}{1-q^{2j-1}}$$

$$= \frac{((1-a)q;q^2)_\infty}{(q;q^2)_\infty} = ((1-a)q;q^2)_\infty\,(-q)_\infty.$$

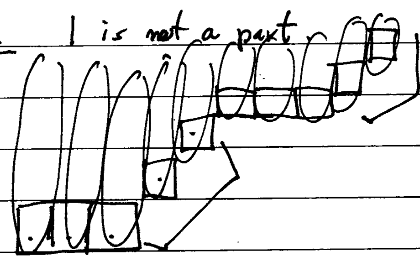
Let λ be a partition into distinct parts (27)
with exactly k runs of one or more consecutive parts.

Case 1 1 is a part of λ.



λ' has unique largest part.
In ① λ' has consecutive parts
Between groups of consecutive parts λ' has repeated parts.
Hence each part $\leq$ largest part appears in λ'
and exactly $(k-1)$ parts appear more than once.

Case 2 1 is not a part



Largest part of λ' is repeated.
again each part $\leq$ largest part of λ' appears in λ'
and exactly k parts appear more that once.

(28)

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B_k(n)\, a^k q^n$$

largest part part of a consecutive group

Case 1

$$= 1 + \sum_{N=1}^{\infty} a q^{N} \left(q^{N-1} + a q^{2(N-1)} + a q^{3(N-1)} + \cdots \right)$$

$$\cdot \left(q^{N-2} + a q^{2(N-2)} + \cdots \right)$$

$$\vdots$$

$$\cdot \left(q + a q^2 + a q^3 + \cdots \right)$$

Case 2

$$+ \sum_{N=1}^{\infty} \left(a q^{2N} + a q^{3N} + \cdots \right) \cdot \prod_{j=1}^{N-1} \left(q^j + a q^{2j} + a q^{3j} + \cdots \right)$$

$$= 1 + \sum_{N=1}^{\infty} a q^N \prod_{j=1}^{N-1} q^j \left(1 + \frac{a q^j}{1-q^j} \right)$$

$$+ \sum_{N=1}^{\infty} \frac{a q^{2N}}{1-q^N} \prod_{j=1}^{N-1} q^j \frac{(1-(1-a)q^j)}{(1-q^j)}$$

$$= 1 + \sum_{N=1}^{\infty} a \left(\frac{(1-q^N)q^N + q^{2N}}{(1-q^N)} \right) q^{N(N-1)/2} \frac{((1-a)q)_{N-1}}{\prod_{j=1}^{N-1}(1-q^j)}$$

$$= 1 + \sum_{N=1}^{\infty} \frac{(1-(1-a))((1-a)q)_{N-1}\, q^{N(N+1)/2}}{(q)_N}$$

$$= 1 + \sum_{N=1}^{\infty} \frac{((1-a)q)_N}{(q)_N} q^{N(N+1)/2}$$

$$= ((1-a)q; q^2)_\infty (-q)_\infty \quad \text{(by Lebesgue Identity)}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_k(n)\, a^k q^n.$$ Hence $A_k(n) = B_k(n)$ for all n. $\square$

Another Refinement of Euler's Theorem

Theorem (N.J. Fine) (1948)

The number of partitions of n into distinct parts with largest part k = The number of partitions of n into odd parts such that $2k+1$ equals the largest part plus twice the number of parts

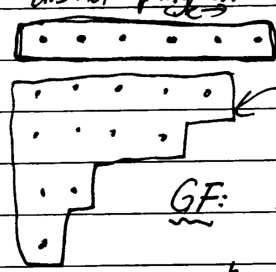
$\ell + 2\# = 21$

Example $n = 19$, $k = 10$ $\qquad 2k+1 = 21$

$10+9$	19	$\ell = 19 \quad \# = 1$
$10+8+1$	$15+3+1$	$\ell = 15 \quad \# = 3$
$10+7+2$	$11+5+1+1+1$	$\ell = 11 \quad \# = 5$
$10+6+3$	$11+3+3+1+1$	
$10+6+2+1$	$7+7+1+1+1+1+1$	$\ell = 7 \quad \# = 7$
$10+5+4$	$7+5+3+1+1+1+1$	
$10+5+3+1$	$7+3+3+3+1+1+1$	
$10+4+3+2$	$3+3+3+3+3+1+1+1+1$	

Proof

pTNs into distinct parts with largest part k:

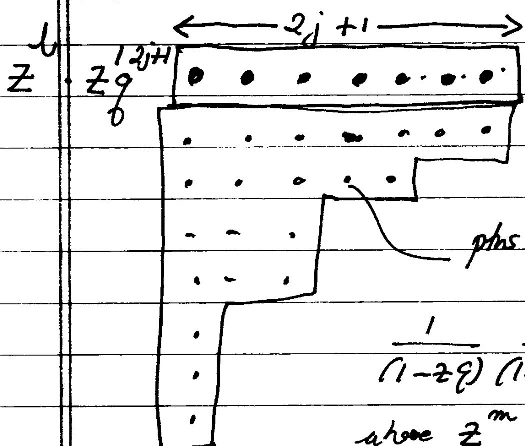


ptns into distinct parts with parts $\leq k-1$.

GF: $z^k q^k (1+q)(1+q^2)\cdots(1+q^{k-1})$

$= z^k q^k (-q)_{k-1}$

PTNS into ODD PARTS :



ptns into odd parts $\le 2j+1$

$$\frac{1}{(1-zq)}\,\frac{1}{(1-zq^3)}\cdots\frac{1}{(1-zq^{2j+1})}$$

where z^m cor. to # of parts m

~~we want~~ $\dfrac{zq^{2j+1}}{(zq;q^2)_{j+1}} = \displaystyle\sum_{\lambda \text{ odd parts } \le 2j+1} z^{\#(\lambda)} q^{|\lambda|}$

We want ~~$l+2\#(\lambda) = 2k+1$~~,

ie ~~$2k = l+1+2\#(\lambda)$~~

Coeff of z^k to generate the desired partitions;

ie $k = l + \#(\lambda)$

$2k = 2l + 2\#(\lambda)$

$2k+1 = 2l+1 + 2\#(\lambda)$

But $2k+1 =$ largest part $+\ 2\#(\lambda)$

$2k+1 = 2j+1 + 2\#(\lambda)$

& hence $l=j$.

Hence G.F. for ptns into odd parts with largest part $2j+1$ in which $2k+1 = 2j+1+2\#(\lambda)$ is coeff of z^k in $z^{j+1}q^{2j+1}/(zq;q^2)_{j+1}$

31/ (~~33~~)

We wish to show that

$$q^k(-q)_{k-1} = \text{coeff of } z^k \text{ in } \sum_{j=0}^{\infty} \frac{z^{j+1} q^{2j+1}}{(zq;q^2)_{j+1}}$$

$$\leftrightarrow \quad \sum_{k=1}^{\infty} z^k q^k (-q)_{k-1} = \sum_{j=0}^{\infty} \frac{z^{j+1} q^{2j+1}}{(zq;q^2)_{j+1}}$$

$$\cancel{\sum_{j=0}^{\infty} \frac{z^{j+1} q^{j+1}}{(zq;q^2)_{j+1}} =}$$

$$\sum_{j=1}^{\infty} z^j q^j (-q)_{j-1} = \sum_{j=0}^{\infty} z^{j+1} q^{j+1} (-q)_j$$

$$= zq \sum_{j=0}^{\infty} z^j q^j (-q)_j$$

$$= zq \sum_{j=0}^{\infty} \frac{z^j q^j (-q)_j (q)_j}{(q)_j}$$

$$= zq \sum_{j=0}^{\infty} z^j q^j \frac{(q^2;q^2)_j}{(q)_j} \frac{(q^{2j+2};q^2)_\infty}{(q^{2j+2};q^2)_\infty}$$

$$= zq(q^2;q^2)_\infty \sum_{j=0}^{\infty} \frac{z^j q^j}{(q)_j} \cdot \frac{1}{(q^{2j+2};q^2)_\infty}$$

$$= zq(q^2;q^2)_\infty \sum_{j=0}^{\infty} \frac{z^j q^j}{(q)_j} \sum_{m=0}^{\infty} \frac{(q^{2j+2})^m}{(q^2;q^2)_m}$$

(32)

$$= zq\,(q^2;q^2)_\infty \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{z^j q^j q^{2mj+2m}}{(q)_j\,(q^2;q^2)_m}$$

$$= zq\,(q^2;q^2)_\infty \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} \frac{q^{2m}}{(q^2;q^2)_m}\,\frac{z^j q^{j(2m+1)}}{(q)_j}$$

$$= zq\,(q^2;q^2)_\infty \sum_{m=0}^{\infty} \frac{q^{2m}}{(q^2;q^2)_m} \sum_{j=0}^{\infty} \frac{z^j q^{j(2m+1)}}{(q)_j}$$

$$= zq\,(q^2;q^2)_\infty \sum_{m=0}^{\infty} \frac{q^{2m}}{(q^2;q^2)_m}\,\frac{1}{(zq^{2m+1})_\infty}\,\frac{(zq)_{2m}}{(zq)_{2m}}$$

$$= \frac{zq\,(q^2;q^2)_\infty}{(zq)_\infty} \sum_{m=0}^{\infty} \frac{q^{2m}}{(q^2;q^2)_m}\cdot (zq)_{2m}$$

$$= \frac{zq\,(q^2;q^2)_\infty}{(zq)_\infty} \sum_{m=0}^{\infty} \frac{(zq;q^2)_m\,(zq^2;q^2)_m\,q^{2m}}{(q^2;q^2)_m}$$

$$= \frac{zq\,(q^2;q^2)_\infty}{(zq)_\infty}\,\frac{(zq^2;q^2)_\infty\,(zq^3;q^2)_\infty}{(q^2;q^2)_\infty} \cdot \sum_{m=0}^{\infty} \frac{(q^2;q^2)_m\,z^m q^{2m}}{(zq^3;q^2)_m\,(q^2;q^2)_m}$$

(33)

$$= \frac{1}{(1-zq)} \sum_{m=0}^{\infty} \frac{z^{m+1} q^{2m+1}}{(zq^3;q^2)_m}$$

$$= \sum_{m=0}^{\infty} \frac{z^{m+1} q^{2m+1}}{(zq;q^2)_{m+1}} \qquad \square$$