$$\left(\frac{1}{\alpha}\right)_n \alpha^n = \left(1-\frac{1}{\alpha}\right)\left(1-q/\alpha\right)\cdots\left(1-q^{n-1}/\alpha\right)\alpha^n \quad (15)$$

$$= (\alpha-1)(\alpha-q)\cdots\cdots(\alpha-q^{n-1})$$

So $\displaystyle\lim_{\alpha\to 0}\left(\frac{1}{\alpha}\right)_n \alpha^n = (-1)^n q^{n(n-1)/2}$.

It follows that

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{\beta}\right)_n \beta^n z^n q^{n(n-1)/2}}{(z)_n (q)_n} = \frac{(z\beta)_\infty}{(z)_\infty}.$$

If we let $\beta \to 0$ we find that

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} (-1)^n q^{n(n-1)/2} z^n}{(z)_n (q)_n} = \frac{1}{(z)_\infty}$$

$$\& \quad \sum_{n=0}^{\infty} \frac{z^n q^{n^2-n}}{(z)_n (q)_n} = \frac{1}{(z)_\infty}.$$

(2) follows from (1) by letting $z=q$. $\square$

COMBINATORIAL PROOF of (1). Replacing z by zq we prove the equivalent statement that

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q)_n (zq)_n} = \frac{1}{(zq)_\infty}.$$

Recall that for $|z|<1$, $|zq|<1$ we have

$$\sum_{\lambda\in\mathcal{P}} z^{\#(\lambda)} q^{|\lambda|} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p(m,n) z^m q^n = \frac{1}{(zq)_\infty}.$$