Proof: Suppose $|q|<1$ & $|q|<|z|$.
We have

$$\sum_{n=0}^{\infty} \frac{z^n q^{\frac{1}{2}n(n-1)}}{(q)_n} = (-z;q)_\infty$$

(by Euler's Cor. to q-bin). So

$$\sum_{n=0}^{\infty} \frac{z^n q^{n(n+1)}}{(q^2;q^2)_n} = (-zq^2;q^2)_\infty.$$

Replace z by zq^{-1} & we have

$$\sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n} = (-zq;q^2)_\infty.$$

So

$$(-zq;q^2)_\infty = \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2;q^2)_n} \qquad (*)$$

$$= \sum_{n=0}^{\infty} \frac{z^n q^{n^2}(q^{2n+2};q^2)_\infty}{(q^2;q^2)_n(q^{2n+2};q^2)_\infty}$$

$$= \sum_{n=0}^{\infty} \frac{z^n q^{n^2}(q^{2n+2};q^2)_\infty}{(q^2;q^2)_\infty}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} z^n q^{n^2}(q^{2n+2};q^2)_\infty$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n^2}(q^{2n+2};q^2)_\infty$$

(since $(q^{2n+2};q^2)_\infty = (1-q^{2n+2})\cdots(1-q^0)\cdots = 0$ if $n<0$)