$$= \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n^2} \sum_{m=0}^{\infty} (-1)^m q^{m(2n+1)} \frac{q^{m^2}}{(q^2;q^2)_m}$$

(by (*) with $z = -q^{2n+1}$)

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^m z^n \frac{q^{m^2+2nm+m+n^2}}{(q^2;q^2)_m}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^m z^n \frac{q^{(m+n)^2+m}}{(q^2;q^2)_m}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{m=0}^{\infty} (-1)^m \frac{q^m z^{-m}}{(q^2;q^2)_m} \sum_{n=-\infty}^{\infty} z^{m+n} q^{(m+n)^2}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{m=0}^{\infty} (-1)^m \frac{q^m z^{-m}}{(q^2;q^2)_m} \sum_{k=-\infty}^{\infty} z^k q^{k^2}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{k=-\infty}^{\infty} z^k q^{k^2} \sum_{m=0}^{\infty} \frac{(-q/z)^m}{(q^2;q^2)_m}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n^2} \frac{1}{(-q/z;q^2)_\infty}$$ (by Euler's Cor. to q-bin.)

Therefore, $$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty$$

for $\left|\frac{q}{z}\right| < 1$. Result holds for ~~$z \neq 0$~~ by analytic continuation. $\square$