(2)

Then
$$\sum_{\lambda \in \mathcal{P}_k} z^{\#(\lambda)} q^{|\lambda|}$$

$$= \sum_{a_1, a_2, \ldots, a_k \geq 0} z^{a_1+a_2+\cdots+a_k} q^{a_1+2a_2+\cdots+ka_k}$$

$$= \left(\sum_{a_1 \geq 0} z^{a_1} q^{a_1}\right)\left(\sum_{a_2 \geq 0} z^{a_2} q^{2a_2}\right)\cdots\left(\sum_{a_k \geq 0} z^{a_k} q^{ka_k}\right)$$

$$= (1 + zq^1 + z^2 q^{1+1} + \cdots)$$
$$(1 + zq^2 + z^2 q^{2+2} + \cdots)$$
$$\vdots$$
$$(1 + zq^k + z^2 q^{k+k} + \cdots)$$

$$= \left(\frac{1}{1-zq}\right)\left(\frac{1}{1-zq^2}\right)\cdots\left(\frac{1}{1-zq^k}\right)$$

provided $|zq| < 1$

Letting $k \to \infty$:
$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p(m,n) z^m q^n = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} p(m,n) z^m\right) q^n$$

$$= \prod_{k=1}^{\infty} \frac{1}{1-zq^k}$$ if $|q| < 1$ & $|zq| < 1$ ie ∎.