Suppose $0<Q<1$ & $Z\neq 0$.
Let $q=\sqrt{Q}=Q^{1/2}$, $z=ZQ^{1/2}$. By J.T.P. we have

$$\sum_{n=-\infty}^{\infty} Z^n Q^{n/2} Q^{n^2/2} = (-ZQ^{1/2}\cdot Q^{1/2};Q)_\infty(-Z^{-1}Q^{-1/2}Q^{1/2};Q)_\infty (Q;Q)_\infty$$

and

$$\sum_{n=-\infty}^{\infty} Z^n Q^{n(n+1)/2} = (-ZQ;Q)_\infty(-Z^{-1};Q)_\infty(Q;Q)_\infty.$$

By analytic continuation in q we have

$$(*)\qquad \sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2} = (-zq;q)_\infty(-z^{-1};q)_\infty(q;q)_\infty$$

for $|q|<1$.

<u>COR</u> (Euler's P.N.T).
For $|q|<1$,

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}.$$

<u>Proof</u>: In $(*)$, replace q by q^3 and let $z=-q^{-1}$.
So
$-zq \to (-)(-q^{-1})q^3 = q^2$
$-z^{-1} \to q$ &

$$\sum_{n=-\infty}^{\infty}(-q)^{-n} q^{\frac{1}{2}(3n^2+3n)} = (q^2;q^3)_\infty(q;q^3)_\infty(q^3;q^3)_\infty$$

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}(3n^2-n)} = \prod_{n=0}^{\infty}(1-q^{3n+2})(1-q^{3n+1})(1-q^{3n+3})$$

Replacing n by $-n$ in the sum we have

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}(3n^2+n)} = \prod_{n=1}^{\infty}(1-q^n). \quad \square$$