COR: (Gauss) For $|q|<1$,

(1) $$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{m=1}^{\infty} \frac{(1-q^m)}{(1+q^m)} = \prod_{m=1}^{\infty} \frac{(1-q^m)^2}{(1-q^{2m})}.$$

(2) $$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \prod_{m=1}^{\infty} \frac{(1-q^{2m})}{(1-q^{2m-1})} = \prod_{m=1}^{\infty} \frac{(1-q^{2m})^2}{(1-q^m)}$$

Proof. In JTP let $z=-1$ we obtain

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = (q;q^2)_\infty (q;q^2)_\infty (q^2;q^2)_\infty$$

$$= (q;q^2)_\infty (q)_\infty$$

$$= \frac{(q)_\infty}{(q^2;q^2)_\infty} (q)_\infty$$

$$= \prod_{m=1}^{\infty} \frac{(1-q^m)^2}{(1-q^{2m})} = \prod_{m=1}^{\infty} \frac{(1-q^m)^2}{(1-q^m)(1+q^m)}$$

$$= \prod_{m=1}^{\infty} \frac{(1-q^m)}{(1+q^m)}.$$

Now $$\sum_{n=-\infty}^{\infty} q^{n(n+1)/2} = \sum_{n=0}^{\infty} q^{n(n+1)/2} + \sum_{n=-1}^{\infty} q^{n(n+1)/2}$$

$$= \sum_{n=0}^{\infty} q^{n(n+1)/2} + \sum_{m=0}^{\infty} q^{(-m-1)(-m)/2}$$ let $m=-n-1$ (ie $n=-m-1$)