$$= \sum_{n=0}^{\infty} q^{n(n+1)/2} + \sum_{m=0}^{\infty} q^{m(m+1)/2}.$$

Thus
$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{1}{2}\sum_{n=-\infty}^{\infty} q^{n(n+1)/2}$$

$$= \frac{1}{2} (-q;q)_\infty (-1;q)_\infty (q;q)_\infty \quad (\text{by } z=1 \text{ in } (*))$$

$$= \frac{1}{2} (-1;q)_\infty (-q;q)_\infty (q;q)_\infty$$

$$= \tfrac{1}{2}(2)(-q;q)_\infty (q^2;q^2)_\infty$$

$$= \prod_{m=1}^{\infty} (1+q^m)(1-q^{2m})$$

$$= \prod_{m=1}^{\infty} \frac{(1-q^{2m})}{(1-q^{2m-1})} = \prod_{m=1}^{\infty} \frac{(1+q^m)(1-q^{2m})(1-q^m)}{(1-q^m)}$$

$$= \prod_{m=1}^{\infty} \frac{(1-q^{2m})^2}{(1-q^m)}. \quad \square$$

<u>COR.</u> (Jacobi) If $|q|<1$, then

$$\prod_{n=1}^{\infty} (1-q^n)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}.$$

<u>Proof</u>: By JTP(*),

$$\sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n(n+1)/2} = (zq)_\infty (z^{-1})_\infty (q)_\infty$$