(23)

$$\sum_{n=-\infty}^{\infty}(-1)^n z^n q^{n(n+1)/2} = \sum_{n=0}^{\infty}(-1)^n z^n q^{n(n+1)/2} + \sum_{n=-1}^{-\infty}(-1)^n z^n q^{n(n+1)/2}$$

$(m=-n-1,\ n=-m-1)$

$$= \sum_{n=0}^{\infty}(-1)^n z^n q^{n(n+1)/2} + \sum_{m=0}^{\infty}(-1)^{-m-1} z^{-m-1} q^{(-m-1)(-m)/2}$$

$$= \sum_{n=0}^{\infty}(-1)^n z^n q^{n(n+1)/2} + \sum_{m=0}^{\infty}(-1)^{m+1} z^{-m-1} q^{m(m+1)/2}$$

$$= \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}(z^n - z^{-n-1})$$

$$= \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}(z^{-n-1})(z^{2n+1}-1), \quad \text{for } z \neq 0.$$

Since

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2} z^{-n-1}(z^{2n+1}-1) = (1-z^{-1})(z^{-1}q)_\infty(zq)_\infty(q)_\infty$$

and

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2} z^{-n-1}\frac{(z^{2n+1}-1)}{(1-z^{-1})} = (z^{-1}q)_\infty(zq)_\infty(q)_\infty \quad \text{for } z \neq 0,1.$$

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2} z^{-n}\frac{(z^{2n+1}-1)}{(z-1)} = (z^{-1}q)_\infty(zq)_\infty(q)_\infty$$

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2} z^{-n}(1+z+z^2+\cdots+z^{2n}) = (z^{-1}q)(zq)_\infty(q)_\infty$$

Letting $z \to 1$ we find that

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} = (q)_\infty^3 = \prod_{n=1}^{\infty}(1-q^n)^3. \quad \square$$