Another Refinement of Euler's Theorem

Theorem (N.J. Fine) (1948)

The number of partitions of n into distinct parts with largest part k = The number of partitions of n into odd parts such that $2k+1$ equals the largest part plus twice the number of parts

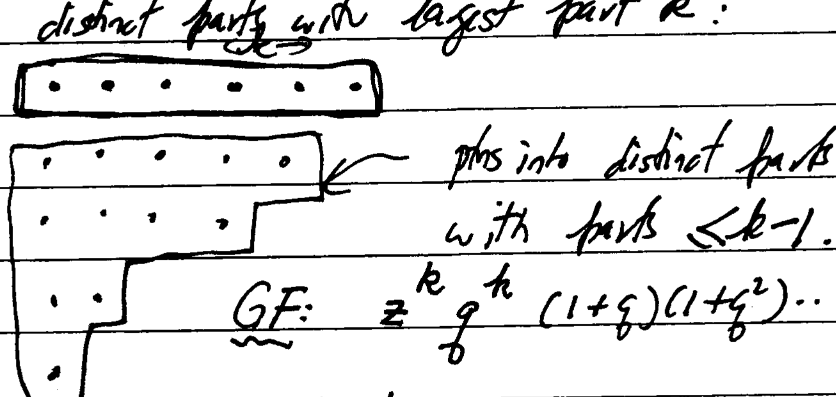
$$\ell + 2\# = 21$$

Example $n = 19$, $k = 10$ $\qquad 2k+1 = 21$

$10+9$	19	$\ell = 19$ $\# = 1$
$10+8+1$	$15+3+1$	$\ell = 15$ $\# = 3$
$10+7+2$	$11+5+1+1+1$	$\ell = 11$ $\# = 5$
$10+6+3$	$11+3+3+1+1$	
$10+6+2+1$	$7+7+1+1+1+1+1$	$\ell = 7$ $\# = 7$
$10+5+4$	$7+5+3+1+1+1+1$	
$10+5+3+1$	$7+3+3+3+1+1+1$	
$10+4+3+2$	$3+3+3+3+3+1+1+1+1$	

Proof

PTNs into distinct parts with largest part k:



GF: $z^k q^k (1+q)(1+q^2)\cdots(1+q^{k-1})$

$$= z^k q^k (-q)_{k-1}$$