Let $H \subset \mathbb{N} = \{1, 2, \ldots\}$.

Let $p(\text{"}H\text{"}, m, n)$ = # of partitions of n with m parts each from H.

Let $p(\text{"}H\text{"}(\leq d), m, n)$ = # of partitions of n with m parts each part from H & each part occurs at most d times.

<u>Theorem</u> Let $|q|<1$ & $|zq|<1$. Then

(i) $$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p(\text{"}H\text{"}, m, n) z^m q^n = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} p(\text{"}H\text{"}, m, n) z^m\right) q^n$$
$$= \prod_{n\in H} (1 - zq^n)^{-1}.$$

(ii) $$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p(\text{"}H\text{"}(\leq d), m, n) z^m q^n$$
$$= \prod_{n\in H} (1 + zq^n + z^2 q^{2n} + \cdots + z^d q^{nd})$$
$$= \prod_{n\in H} \frac{(1 - z^{d+1} q^{(d+1)n})}{(1 - zq^n)}.$$

<u>Notation</u>: Let $q, a \in \mathbb{C}$, $|q|<1$. Let $n \geq 1$ $(n \in \mathbb{Z})$.

$(a)_n := (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}) = \prod_{k=0}^{n-1}(1-aq^k)$,

$(a)_0 := 1$.

$(a)_\infty = (a;q)_\infty = \lim_{n\to\infty} (a;q)_n = \prod_{k=0}^{\infty} (1-aq^k)$.

Hence
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p(m,n) z^m q^n = \prod_{n=1}^{\infty} \frac{1}{1-zq^n} = \frac{1}{(zq;q)_\infty} \quad \text{for } |q|<1,\ |zq|<1.$$