We wish to show that

$$q^k(-q)_{k-1} = \text{coeff of } z^k \text{ in } \sum_{j=0}^{\infty} \frac{z^{j+1} q^{2j+1}}{(zq;q^2)_{j+1}}$$

$$\Leftrightarrow \sum_{k=1}^{\infty} z^k q^k (-q)_{k-1} = \sum_{j=0}^{\infty} \frac{z^{j+1} q^{2j+1}}{(zq;q^2)_{j+1}}$$

$$\cancel{\sum_{j=0}^{\infty} \frac{z^{j+1} q^{j+1}}{(zq;q^2)_{j+1}} =}$$

$$\sum_{j=1}^{\infty} z^j q^j (-q)_{j-1} = \sum_{j=0}^{\infty} z^{j+1} q^{j+1} (-q)_j$$

$$= zq \sum_{j=0}^{\infty} z^j q^j (-q)_j$$

$$= zq \sum_{j=0}^{\infty} \frac{z^j q^j (-q)_j (q)_j}{(q)_j}$$

$$= zq \sum_{j=0}^{\infty} z^j q^j \frac{(q^2;q^2)_j}{(q)_j} \frac{(q^{2j+2};q^2)_\infty}{(q^{2j+2};q^2)_\infty}$$

$$= zq(q^2;q^2)_\infty \sum_{j=0}^{\infty} \frac{z^j q^j}{(q)_j} \cdot \frac{1}{(q^{2j+2};q^2)_\infty}$$

$$= zq(q^2;q^2)_\infty \sum_{j=0}^{\infty} \frac{z^j q^j}{(q)_j} \sum_{m=0}^{\infty} \frac{(q^{2j+2})^m}{(q^2;q^2)_m}$$