(Cauchy) <u>Theorem</u> (q-binomial Theorem). If $|q|<1$ and $|z|<1$ then

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n = \frac{(az)_\infty}{(z)_\infty} = \frac{(1-az)(1-azq)\cdots}{(1-z)(1-zq)\cdots}$$

<u>Proof</u> Let $a, q \in \mathbb{C}$, $|q|<1$.

$$\frac{(az)_\infty}{(z)_\infty} = \prod_{n=0}^{\infty} \frac{(1-azq^n)}{(1-zq^n)}$$ converges uniformly

on compact subsets of $|z|<1$ and defines an analytic function for $|z|<1$.

ie $$F(z) := \frac{(az)_\infty}{(z)_\infty} = \sum_{n=0}^{\infty} A_n z^n \quad \text{for } |z|<1.$$

$$F(zq) = \prod_{n=0}^{\infty} \frac{(1-azq^{n+1})}{(1-zq^{n+1})} = \frac{(1-azq)(1-azq^2)\cdots}{(1-zq)(1-zq^2)\cdots}$$

So

$$\frac{(1-az)}{(1-z)} F(zq) = F(z), \ \&$$

$$(1-az)\, F(zq) = (1-z)\, F(z).$$

$$(1-z)\, F(z) = (1-z) \sum_{n=0}^{\infty} A_n z^n$$

$$= \sum_{n=0}^{\infty} A_n z^n - \sum_{n=0}^{\infty} A_n z^{n+1}$$

$$= \sum_{n=0}^{\infty} A_n z^n - \sum_{n=1}^{\infty} A_{n-1} z^n$$

$$= A_0 + \sum_{n=1}^{\infty} (A_n - A_{n-1}) z^n$$