<u>Corollary (Euler)</u> Suppose $|q|<1$.

(1) $$\sum_{n=0}^{\infty} \frac{z^n}{(q)_n} = \frac{1}{(z)_\infty} \quad \text{for } |z|<1.$$

(2) $$\sum_{n=0}^{\infty} \frac{(-z)^n q^{n(n-1)/2}}{(q)_n} = (z)_\infty \quad \text{for all } z.$$

<u>Proof:</u>

(1) Let $a=0$ in q-bin. Thm gives
$$\sum_{n=0}^{\infty} \frac{z^n}{(q)_n} = \frac{1}{(z)_\infty}.$$

(2) In q-bin. Thm replace a by a/b & z by bz
$$\sum_{n=0}^{\infty} \frac{\left(\frac{a}{b}\right)_n}{(q)_n} \frac{(b^n z^n)}{1} = \frac{(az)_\infty}{(bz)_\infty} \quad \text{provided } |bz|<$$

$$\left(\frac{a}{b}\right)_n b^n = b^n \left(1-\frac{a}{b}\right)\left(1-\frac{a}{b}q\right)\left(1-\frac{a}{b}q^2\right)\cdots\left(1-\frac{a}{b}q^{n-1}\right)$$
$$= (b-a)(b-aq)(b-aq^2)\cdots(b-aq^{n-1})$$

$$\lim_{b\to 0}\left(\frac{a}{b}\right)_n b^n = (-a)(-aq)\cdots(-aq^{n-1})$$
$$= (-1)^n a^n q^{n(n-1)/2}.$$

It can be shown that the series (let z, q, a be fixed, $|q|<1$)
$$\sum_{n=0}^{\infty}\left(\frac{a}{b}\right)_n b^n z^n \quad \text{converges uniformly}$$
in b (for b on some disk $|b|\le\varepsilon$). Hence
$$\lim_{b\to 0}\sum_{n=0}^{\infty}\left(\frac{a}{b}\right)_n b^n z^n = \sum_{n=0}\lim_{b\to 0}\left(\frac{a}{b}\right)_n b^n z^n$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{n(n-1)/2} z^n}{(q)_n}$$