Hence

$$\sum_{\lambda\in\tilde{\mathcal{P}}_k} z^{\#(\lambda)} q^{|\lambda|} = \frac{z^k q^k}{(q)_k}$$

Since

$$\mathcal{P} = \cup\, \tilde{\mathcal{P}}_k \quad (\text{disjoint})$$

$$\sum_{\lambda\in\mathcal{P}} z^{\#(\lambda)} q^{|\lambda|} = 1 + \sum_{k=1}^{\infty} \sum_{\lambda\in\mathcal{P}_k} z^{\#(\lambda)} q^{|\lambda|}$$

Hence $$\frac{1}{(zq)_\infty} = 1 + \sum_{k=1}^{\infty} \frac{z^k q^k}{(q)_k}.$$

<u>Combinatorial Proof of (2)</u> Replacing z by $-zq$ we write (2) in the equivalent form

$$1 + \sum_{k=1}^{\infty} \frac{z^k q^{k(k+1)/2}}{(q)_k} = (-zq)_\infty = \prod_{m=1}^{\infty} (1 + z q^{m})$$

Generating function of partitions into distinct parts

Let $k \geq 1$. Let PD_k = set of partitions into exactly k distinct parts.

(Eg: $k=7$

$\lambda = (11, 10, 8, 5, 4, 2, 1)$ of n

We remove 1 from smallest part

2 from next part