Chapter 3 Ramanujan's Partition Congruences

(See Ch 10 of Text).

In 1919, Ramanujan stated and proved the following congruences:

$$p(5n+4) \equiv 0 \pmod{5}$$
$$p(7n+5) \equiv 0 \pmod{7}$$

and
$$p(11n+6) \equiv 0 \pmod{11},$$

for all $n \geq 0$.

He also conjectured that if $\delta = 5^a, 7^b$ or 11^c and $24\lambda \equiv 1 \pmod{\delta}$. Then

$$p(\delta n + \lambda) \equiv 0 \pmod{\delta}.$$

Not quite correct. Correction made by Chowla (1934)

$$p(7^b n + \lambda_b) \equiv 0 \pmod{7^{[(b+2)/2]}}$$

where $24\lambda_b \equiv 1 \pmod{7^b}$.

Ramanujan's conjecture was proved by Watson (1938) for $\delta = 5^a, 7^b$, and for $\delta = 11^c$ by Atkin (1967).

~~Atkin and O'Brien (1967) found~~

~~$p(11^3 \cdot 13 n +$~~

Atkin (1969) found

$$p(59^4 \cdot 13n + 111247) \equiv 0 \pmod{13},$$
$$p(23^3 \cdot 17n + 2623) \equiv 0 \pmod{17}.$$

Kolberg (1959) proved

$$p(n) \equiv 0 \pmod{2} \quad \text{for infinitely many } n$$

& $$p(n) \equiv 1 \pmod{2} \quad \text{for infinitely many } n$$

(2)

Conjecture:
$$\left|\{n \leq N : p(n) \text{ is even}\}\right| \sim \frac{1}{2} N$$
as $N \to \infty$

~~Conj~~ $\boxed{\text{Serre (1998)}}$ has proved that
$$\lim_{N\to\infty} \frac{\left|\{n \leq N : p(n) \text{ is even}\}\right|}{\sqrt{N}} = +\infty$$

Conjecture: $p(n) \equiv 0 \pmod 3$ for infinitely many n.

$\boxed{\text{Ono (2000)}}$ has proved that for every prime $\ell \geq 5$ there exist infinitely many pairs (A, B) such that
$$p(An + B) \equiv 0 \pmod{\ell}$$
for all $n \geq 0$.

Definition Let $A = \sum_{n=0}^{\infty} a_n q^n$, $B = \sum_{n=0}^{\infty} b_n q^n \in \mathbb{Z}[[q]]$
~~be power series & let m≥1~~
(i.e. power series with integer coefficients)
Let $m \geq 1$. We say
$$A \equiv B \pmod m$$
iff $a_n \equiv b_n \pmod m$ for all $n \geq 1$.
Note ~~Note~~ This is equivalent to existence of a $C = \sum_{n=0}^{\infty} c_n q^n \in \mathbb{Z}[[q]]$ such that $A = mC + B$.

<u>Lemma:</u> Let $A=\sum_{n=0}^{\infty} a_n q^n$, $B=\sum_{n=0}^{\infty} b_n q^n$, (3)

$C=\sum_{n=0}^{\infty} c_n q^n \in \mathbb{Z}[[q]]$, $D \in \sum_{n=0}^{\infty} d_n q^n \in$

Then

(1) if $A \equiv B \pmod{m}$ and $C \equiv D \pmod{m}$

then

(i) $A+C \equiv B+D \pmod{m}$

(ii) $AC \equiv BD \pmod{m}$

(2) If $a_0=1$ then $\frac{1}{A} \in \mathbb{Z}[[q]]$.

(3) If $A \equiv B \pmod{m}$ & $c_0=1$ then

$\frac{A}{C} \equiv \frac{B}{C} \pmod{m}$.

(4) If $a_0=b_0=1$ and $A\equiv B \pmod{m}$ then $\frac{1}{A} \equiv \frac{1}{B} \pmod{m}$

Ramanujan's Congruence $p(5n+4) \equiv 0 \pmod{5}$

<u>Lemma</u> Let p be prime.

Then

(1) $(1-q)^p \equiv (1-q^p) \pmod{p}$.

(2) $(q)_\infty^p \equiv (q^p;q^p)_\infty \pmod{p}$.

<u>Proof</u>:

(1) By Binomial Thm,

$$(1-q)^p = \sum_{j=0}^{p} \binom{p}{j} 1^j (-q)^{p-j}$$

Let $1 \le j \le p-1$, then $\binom{p}{j} \in \mathbb{Z}$ and

$$\binom{p}{j} = \frac{p(p-1)\cdots(p-j+1)}{j(j-1)\cdots(2)(1)},$$

$$j!\binom{p}{j} = p(p-1)\cdots(p-j+1) \equiv 0 \pmod{p}.$$

Since p is prime & $j<p$, $p \nmid j!$.

It follows that $p \mid \binom{p}{j}$ & $\binom{p}{j} \equiv 0 \pmod{p}$.

Hence,

$$
\begin{aligned}
(1-q)^p &\equiv (-q)^p + 1 \pmod{p} \\
&\equiv 1 + (-1)^p q^p \pmod{p} \\
&\equiv \begin{cases} 1 - q^p \pmod{p} & \text{if } p \text{ odd} \\ 1 + q^p \equiv 1 - q^p \pmod{p} & \text{if } p = 2. \end{cases}
\end{aligned}
$$

(2) Let $N \geq 1$.

$$
\begin{aligned}
(q)_\infty^{\,p} &= \left((1-q)(1-q^2)\cdots(1-q^N)\right)^p + O(q^{N+1}) \\
&= (1-q)^p(1-q^2)^p\cdots(1-q^N)^p + O(q^{N+1}) \\
&\equiv (1-q^p)(1-q^{2p})\cdots(1-q^{pN}) + O(q^{N+1}) \pmod{p} \\
&\equiv (q^p;q^p)_N + O(q^{N+1}) \pmod{p} \\
&\equiv (q^p;q^p)_\infty + O(q^{N+1}) \pmod{p}
\end{aligned}
$$

True for all $N \geq 1$. Result follows. $\square$

Theorem (Ramanujan) (5)

$$p(5n+4) \equiv 0 \pmod 5$$

for all $n \geq 0$.

Proof: We need

$$(q)_\infty = \prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}$$

& $$(q)_\infty^3 = \prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n\geq 0} (-1)^n (2n+1) q^{n(n+1)/2}$$

$$\frac{1}{(q)_\infty} = \sum_{n=0}^{\infty} p(n) q^n$$

$$\frac{(q)_\infty^4}{(q)_\infty^5} = \sum_{n=0}^{\infty} p(n) q^n$$

$$(q)_\infty^5 \equiv (q^5;q^5)_\infty \pmod 5$$

so

$$\frac{1}{(q)_\infty^5} \equiv \frac{1}{(q^5;q^5)_\infty} \pmod 5$$

and $$\frac{(q)_\infty^4}{(q)^5} \equiv \frac{(q)_\infty^4}{(q^5;q^5)_\infty} \pmod 5$$

$$\frac{1}{(q^5;q^5)_\infty} = \sum_{n=0}^{\infty} p(n) q^{5n}$$

$$(q)_\infty^4 = (q)_\infty (q)_\infty^3 = \sum_{i=-\infty}^{\infty} (-1)^i q^{i(3i-1)/2} \sum_{j=0}^{\infty} (-1)^j (2j+1) q^{\frac{j(j+1)}{2}}$$

Hence $$\sum_{n=0}^{\infty} p(n) q^n \equiv \sum_{i=-\infty}^{\infty} (-1)^i q^{i(3i-1)/2} \sum_{j=0}^{\infty} (-1)^j (2j+1) q^{j(j+1)/2} \cdot \sum_{k=0}^{\infty} p(k) q^{5k}$$

(6)

Hence

$$p(n) \equiv \sum_{\substack{\frac{i(3i-1)}{2} + \frac{j(j+1)}{2} + 5k = n \\ j \geq 0 \text{ and } k \geq 0}} (-1)^{i+j}(2j+1)\, p(k) \pmod 5$$

$$(*) \quad p(5n+4) \equiv \sum_{\substack{\frac{i(3i-1)}{2} + \frac{j(j+1)}{2} + 5k = 5n+4 \\ j \geq 0 \;\&\; k \geq 0}} (-1)^{i+j}(2j+1)\, p(k) \pmod 5$$

Suppose $\frac{i(3i-1)}{2} + \frac{j(j+1)}{2} + 5k = 5n+4$ where $j \geq 0$ & $k \geq 0$.

Then

$$\frac{i(3i-1)}{2} + \frac{j(j+1)}{2} \equiv 4 \pmod 5$$

$$3i^2 - i + j^2 + j \equiv 8 \equiv 3 \pmod 5$$

$$3(i^2 - 2i) + j^2 - 4j \equiv 3 \pmod 5$$

$$3(i^2 - 2i + 1 - 1) + (j^2 - 4j + 4 - 4) \equiv 3 \pmod 5$$

$$3((i-1)^2 - 1) + (j-2)^2 - 4 \equiv 3 \pmod 5$$

$$3(i-1)^2 + (j-2)^2 \equiv 0 \pmod 5$$

$$(i-1)^2 \equiv 0^2, (\pm 1)^2, (\pm 2)^2 \equiv 0, 1, 4 \pmod 5$$

$$\cancel{(j-2)^2 \equiv 0, 1, 4 \pmod 5}$$

$$3(i-1)^2 \equiv 0, 3, 2 \pmod 5$$

$$(j-2)^2 \equiv 0, 1, 4 \pmod 5$$

$$3(i-1)^2 + (j-2)^2 \equiv 0 \pmod 5$$

iff $3(i-1)^2 \equiv 0$ & $(j-2)^2 \equiv 0 \pmod 5$

ie $i \equiv 1$ & $j \equiv 2 \pmod 5$

Hence $(2j+1) \equiv 4+1 \equiv 0 \pmod 5$ & by $(*)$

$$p(5n+4) \equiv 0 \pmod 5. \quad \blacksquare$$

Roots of Unity

Let $\theta \in \mathbb{R}$.

$$e^{i\theta} := \cos\theta + i\sin\theta.$$

Then $e^{i(\theta_1+\theta_2)} = e^{i\theta_1} e^{i\theta_2}$

and $(e^{in\theta}) = (e^{i\theta})^n$ (de Moivre)

$$\cos n\theta + i\sin n\theta = (\cos(\theta) + i\sin\theta)^n,$$

for $n \in \mathbb{Z}$.

Theorem Let n be a positive integer. The equation $z^n = 1$

has n complex roots

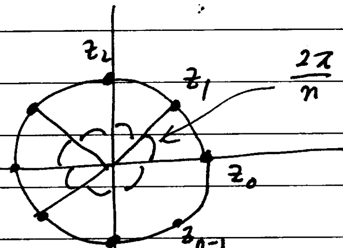
$$z_k = e^{2\pi i \frac{k}{n}} \quad k = 0, 1, 2, \dots, n-1$$

ie $z_k = \zeta^k$, $k = 0, 1, 2, \dots, n-1$

where $\zeta = e^{2\pi i/n} = \cos(\pi/n) + i\sin(\pi/n)$.

Further, $1 + \zeta + \dots + \zeta^{n-1} = 1$

and $(z^n - 1) = (z-1)(z-\zeta)(z-\zeta^2)\cdots(z-\zeta^{n-1})$.



Proof

Let $z_k = e^{2\pi i k/n}$ $0 \le k \le n-1$.

$$z_k^n = e^{2\pi i k \frac{n}{n}} = e^{2\pi i k}$$
$$= \cos(2\pi k) + i\sin(2\pi k)$$
$$= \cos(0) + i\sin(0) = 1.$$

It can be shown that the z_k, $0 \le k \le n-1$ are distinct and account for all the roots. It follows that

$$(z^n - 1) = (z-1)(z-\zeta)\cdots(z-\zeta^{n-1}).$$

Also $(z^n - 1) = (z-1)(1 + z + \dots + z^{n-1})$.

$$0 = (\zeta^n - 1) = (\zeta - 1)(1 + \zeta + \dots + \zeta^{n-1}).$$

(8)

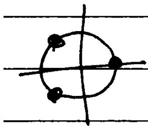
Since $\zeta \neq 1$, we have

$$1+\zeta+\zeta^2+\cdots+\zeta^{n-1}=0. \quad \square$$

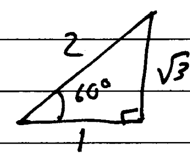
Note: ζ is called an n^{th} root of unity.

Example: Solve $z^3=1$.

$$z_k = \cos\left(\frac{2\pi k}{3}\right)+i\sin\left(\frac{2\pi k}{3}\right), \quad k=0,1,2.$$



$$z_0 = 1$$



$$\begin{aligned} z_1 &= \cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right) \\ &= \cos(120^\circ)+i\sin(120^\circ) \\ &= \cos(180^\circ-60^\circ)+i\sin(180^\circ-60^\circ) \\ &= -\cos 60^\circ + i\sin 60^\circ \\ &= -1/2 + i\sqrt{3}/2 \end{aligned}$$

$$\begin{aligned} z_2 &= \cos(240^\circ)+i\sin(240^\circ) \\ &= -1/2 - i\sqrt{3}/2 \end{aligned}$$

Theorem (Ramanujan)

$$\sum_{n=0}^{\infty} p(5n+4)\,q^n = 5\prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)^6}$$

Proof:

$$(q)_\infty = \prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}$$

n	$n(3n-1)/2 \pmod 5$
0	0
1	$1(2)/2 \equiv 1$
2	$2(5)/2 \equiv 0$
3	$3(8)/2 \equiv 12 \equiv 2$
4	$(-1)(-4)/2 \equiv 2$

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$n = 5k+r \quad (r=0,1,2,3,4).$

$$\frac{n(3n-1)}{2} = \frac{75}{2}k^2 + 15kr - \frac{5}{2}k + \frac{3}{2}r^2 - \frac{1}{2}r$$

$n = 5k \qquad \frac{n(3n-1)}{2} = \frac{75}{2}k^2 - \frac{5}{2}k = \frac{5}{2}k(15k-1)$

$n = 5k+1 \qquad \frac{n(3n-1)}{2} \quad \frac{75k^2}{2} + \frac{25k}{2} + 1 = 25\,\frac{k(3k+1)}{2} + 1$

$n = 5k+2 \qquad \frac{n(3n-1)}{2} = \frac{75k^2}{2} + \frac{55k}{2} + 5 = \frac{5}{2}k(15k+11) + 5$

$n = 5k+3 \qquad \frac{n(3n-1)}{2} = \frac{75k^2}{2} + \frac{85k}{2} + 12 = \frac{5k}{2}(15k+17) + 12$

$n = 5k \cancel{+4}\,{-1} \qquad \frac{n(3n-1)}{2} = \frac{75k^2}{2} - \frac{35k}{2} + 2 = \frac{5k}{2}(15k-7) + 2$

$$(q)_\infty = \sum_k (-1)^{5k} q^{\frac{5k}{2}(15k-1)} + \sum_k (-1)^{5k+2} q^{\frac{5k}{2}(15k+11)+5}$$

$$+ \sum_k (-1)^{5k+1} q^{25k(3k+1)/2+1}$$

$$+ \sum_k (-1)^{5k+3} q^{\frac{5k}{2}(15k+17)+12} + \sum_k (-1)^{5k-1} q^{\frac{5k}{2}(15k-7)+2}$$

$$= j_0(q^5) - q \sum_k (-1)^k (q^{25})^{k(3k+1)/2} + q^2 j_2(q^5)$$

$$(q)_\infty = j_0(q^5) - q\,(q^{25};q^{25})_\infty + q^2 j_2(q^5)$$

$$\frac{(q)_\infty}{(q^{25};q^{25})_\infty} = J_0(q^5) - q + q^2 J_2(q^5)$$

where $J_0(q^5) = \dfrac{j_0(q^5)}{(q^{25};q^{25})_\infty}$, $J_2(q^5) = \dfrac{j_2(q^5)}{(q^{25};q^{25})_\infty}$.

$$\frac{(q)_\infty^3}{(q^{25};q^{25})_\infty^3} = \left[J_0 - q + q^2 J_2\right]^3 \qquad (10)$$

$$= \left(J_0^3 - 3q^5 J_2^2\right) + \left(-3J_0^2 q + q^6 J_2^3\right)$$

$$+ \left(3q^2 J_0^2 J_2 + 3J_0 q^2\right) + \left(-6q^3 J_0 J_2 - q^3\right)$$

$$+ \left(3J_0 q^4 J_2^2 + 3q^4 J_2\right)$$

$$= \left(J_0^3 - 3q^5 J_2^2\right) + q\left(q^5 J_2^3 - 3J_0^2\right)$$

$$+ 3q^2 (J_0)(J_0 J_2 + 1) - q^3 (1 + 6J_0 J_2)$$

$$+ 3J_2 q^4 (J_0 J_2 + 1).$$

Now $(q)_\infty^3 = \sum_{n \geq 0} (-1)^n (2n+1) q^{n(n+1)/2}$.

n	$n(n+1)/2 \pmod 5$
0	0
1	1
2	3
3	$6 \equiv 1$
4	0

Hence ~~q~~ The powers series expansion of $(q)_\infty^3$ does not have terms involving q^{5n+2} & q^{5n+4}.

It follows that the power series expansion (11)

of $\dfrac{(q)_\infty^3}{(q^{25};q^{25})_\infty^3}$ does not have terms involving q^{5n+2}, q^{5n+4}.

Hence

$$J_0 J_2 + 1 = 0$$

and

$$J_0 J_2 = -1,$$

$$J_2 = -\frac{1}{J_0}.$$

Hence

$$\frac{(q)_\infty}{(q^{25};q^{25})_\infty} = J_0(q^5) - q - \frac{q^2}{J_0(q^5)}$$

and

$$\frac{1}{(q)_\infty} = (q^{25};q^{25})_\infty \times \frac{1}{J_0(q^5) - q - \dfrac{q^2}{J_0(q^5)}}$$

Let $\zeta = e^{2\pi i/5}$. Then

$$z^5 - 1 = (z-1)(z-\zeta)(z-\zeta^2)(z-\zeta^3)(z-\zeta^4)$$

$$= (-1)(1-z)(-\zeta)(1-\zeta^4 z)(-\zeta^2)(1-\zeta^3 z)$$

$$(-\zeta^3)(1-\zeta^2 z)(-\zeta^4)(1-\zeta z), \text{ since } \zeta^5 = 1$$

$$= (-1)^5 \zeta^{1+2+3+4} (1-z)(1-\zeta z)(1-\zeta^2 z)(1-\zeta^3 z)(1-\zeta^4 z)$$

$$z^5 - 1 = -\zeta^{10} (1-z)(1-\zeta z)(1-\zeta^2 z)(1-\zeta^3 z)(1-\zeta^4 z)$$

&

$$(1-z)(1-\zeta z)(1-\zeta^2 z)(1-\zeta^3 z)(1-\zeta^4 z) = 1 - z^5$$

$$\prod_{j=0}^{4} (1-\zeta^j z) = (1-z^5)$$

Claim: $$\prod_{j=0}^{4}\left(1-(\zeta^k)^j z\right) = \begin{cases} (1-z^5) & 1\le k\le 4 \\ (1-z)^5 & k=0 \end{cases} \qquad (12)$$

~~for k=1,2,3,4.~~

(EX)

Let $E(q)=(q)_\infty = \prod_{n=0}^{\infty}(1-q^{5n+5})(1-q^{5n+1})(1-q^{5n+2})(1-q^{5n+3})(1-q^{5n+4})$

$$= \prod_{n=0}^{\infty}(1-q^{5n+5}) \prod_{k=1}^{4}\prod_{n=0}^{\infty}(1-q^{5n+k})$$

$$E(\zeta^l q) = \prod_{n=0}^{\infty}\left(1-(\zeta^l q)^{5n+5}\right) \prod_{n=0}^{\infty}\prod_{k=1}^{4}\left(1-(\zeta^l q)^{5n+k}\right)$$

$$= \prod_{n=0}^{\infty}(1-q^{5n+5}) \prod_{n=0}^{\infty}\prod_{k=1}^{4}\left(1-(\zeta^k)^l q^{5n+k}\right)$$

$$E(q)\,E(\zeta q)\,E(\zeta^2 q)\,E(\zeta^3 q)\,E(\zeta^4 q)$$

$$= \prod_{l=0}^{4} E(\zeta^l q)$$

$$= \prod_{n=0}^{\infty}(1-q^{5n+5})^5 \prod_{n=0}^{\infty}\prod_{k=1}^{4}\prod_{l=0}^{4}\left(1-q^{5n+k}(\zeta^k)^l\right)$$

$$= (q^5;q^5)_\infty^5 \prod_{n=0}^{\infty}\prod_{k=1}^{4}\left(1-q^{5(5n+k)}\right)$$

$$= (q^5;q^5)_\infty^5\,(q^5;q^{25})_\infty(q^{10};q^{25})_\infty(q^{15};q^{25})_\infty(q^{20};q^{25})_\infty \times \frac{(q^{25};q^{25})_\infty}{(q^{25};q^{25})_\infty}$$

$$= \frac{(q^5;q^5)_\infty^6}{(q^{25};q^{25})_\infty} = \frac{E(q^5)^6}{E(q^{25})}$$

(13)

now

$$E(q) = E(q^{25})\left[J_0(q^5) - q - \frac{q^2}{J_0(q^5)} \right]$$

$$\cancel{E(\zeta q)E(\zeta^2 q)E(\zeta^3 q)E(\zeta^4 q)}$$

$$\Rightarrow \quad E(\zeta^j q) = E((q\zeta^j)^{25})\left[J_0((\zeta^j q)^5) - \zeta^j q - \frac{\zeta^{2j} q^2}{J_0((\zeta^j q)^5)} \right]$$

$$= E(q^{25})\left[J_0(q^5) - \zeta^j q - \frac{\zeta^{2j} q^2}{J_0(q^5)} \right]$$

since $\zeta^{5j} = (\zeta^5)^j = 1.$

$$E(\zeta q)E(\zeta^2 q)E(\zeta^3 q)E(\zeta^4 q)$$

$$= [E(q^{25})]^4 \left[J_0 - \zeta q - \frac{\zeta^2 q^2}{J_0} \right]$$

$$\times \left[J_0 - \zeta^2 q - \frac{\zeta^4 q^2}{J_0} \right] \times \left[J_0 - \zeta^3 q - \frac{\zeta^6 q^2}{J_0} \right]$$

$$\times \left[J_0 - \zeta^4 q - \frac{\zeta^8 q^2}{J_0} \right], \quad \text{where } J_0 = J(q^5)$$

$$= \Big[(J_0^4 - 3q^5 J_0^{-1}) + qJ_0^3 + 2q^6 J_0^{-2})$$

$$+ (2q^2 J_0^2 - q^7 J_0^{-2}) + (3q^3 J_0 + q^8 J_0^{-4})$$

$$+ 5q^4 \Big] [E(q^{25})]^4$$

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{E(q)} = \frac{E(\zeta q)\,E(\zeta^2 q)\,E(\zeta^3 q)\,E(\zeta^4 q)}{E(q)\,E(\zeta q)\,E(\zeta^2 q)\,E(\zeta^3 q)\,E(\zeta^4 q)} \tag{14}$$

$$= \left(\frac{E(q^{25})}{E(q^5)^6}\right)\left(E(q^{25})\right)^4$$

$$\Big[(J_0^4 - 3q^5 J_0^{-1}) + q(J_0^3 + 2q^5 J_0^{-2}) + q^2(2J_0^2 - q^5 J_0^{-2}) + q^3(3J_0 + q^5 J_0^{-4}) + 5q^4 \Big]$$

Hence

$$\sum_{n=0}^{\infty} p(5n+4) q^{5n+4} = 5q^4 \frac{E(q^{25})^5}{E(q^5)^6},$$

$$\sum_{n=0}^{\infty} p(5n+4) q^{5n} = 5\,\frac{E(q^{25})^5}{E(q^5)^6}$$

$$\sum_{n=0}^{\infty} p(5n+4) q^{n} = \frac{5\,E(q^5)^5}{E(q)^6} = 5\,\frac{(q^5;q^5)_\infty^5}{(q)_\infty^6}$$

$$= 5 \prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)^6}. \qquad \Box$$

(15)

<u>Winquist's Identity</u> (1969)

Suppose $a, b \neq 0$ and $|q|<1$. Then

$$(a)_\infty (q/a)_\infty (b)_\infty (q/b)_\infty (\tfrac{a}{b})_\infty (\tfrac{bq}{a})_\infty$$
$$(ab)_\infty (\tfrac{q}{ab})_\infty (q)_\infty^2$$

$$= \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} (-1)^{i+j} \Big((a^{-3i} - a^{3i+3})(b^{-3j} - b^{3j+1})$$
$$+ (a^{-3j+1} - a^{3j+2})(b^{3i+2} - b^{-3i-1}) \Big)$$
$$\cdot q^{\frac{3}{2} i(i+1) + j(3j+1)/2}.$$

<u>Proof</u>:

Let

$$F(a,b,q) := (a)_\infty (q/a)_\infty (b)_\infty (q/b)_\infty (\tfrac{a}{b})_\infty (\tfrac{bq}{a})_\infty$$
$$(ab)_\infty (\tfrac{q}{ab})_\infty (q)_\infty^2.$$

Let $f(a) = (a)_\infty (q/a)_\infty$

$$f(aq) = (aq)_\infty (1/a)_\infty = f(\tfrac{1}{a})$$
$$= f(a) \cdot \frac{(1-a^{-1})}{(1-a)} = \frac{-a^{-1}(1-a)}{(1-a)} f(a)$$
$$= -a^{-1} f(a).$$

$$F(a,b,q) := f(a) f(b) f(\tfrac{a}{b}) f(ab) \quad (q)_\infty^2$$

$$F(aq,b,q) := f(aq) f(b) f(\tfrac{aq}{b}) f(abq) \quad (q)_\infty^2$$
$$= (-a^{-1}) f(a) f(b) (-\tfrac{a}{b})^{-1} f(\tfrac{a}{b}) (-ab)^{-1} f(ab) \, (q)_\infty^2$$

$$= -a^{-3} F(a,b,q).$$

(16)

$$F\left(\tfrac{1}{a}, b, q\right) = f\left(\tfrac{1}{a}\right) f(b)\, f\left(\tfrac{1}{ab}\right) f\left(\tfrac{b}{a}\right) (q)_\infty^2$$

$$= \left(-a^{-1}\right) f(a) f(b) (-ab)^{-1} f(ab) \left(-\tfrac{a}{b}\right)^{-1} f\left(\tfrac{a}{b}\right) (q)_\infty^2$$

$$= \left(-a^{-3}\right) F(a, b, q).$$

It can be shown that $F(a,b,q)$ has a Laurent series expansion

$$F(a,b,q) := \sum_{n=-\infty}^{\infty} A_n(b,q)\, a^n \qquad (0<|a|<\infty)$$

$$F(aq,b,q) = \sum_n A_n(b,q)\, a^n q^n =$$

$$= -a^{-3} F(a,b,q) = \sum_n -A_n a^{n-3}$$

$$= \sum_n -A_{n+3} a^n \quad \&$$

$$\sum_n A_n a^n q^n = -\sum_n A_{n+3} a^n$$

Therefore $\quad A_{n+3} = -A_n q^n$

Also,

$$F\left(\tfrac{1}{a}, b, q\right) = F(aq, b, q)$$

$$\sum_n A_{-n} a^n = \sum A_n a^{-n} = \sum A_n a^n q^n$$

$$A_{-n} = q^n A_n.$$

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$$A_2 = -A_{-1}\, q^{-1} = -(q A_1)\, q^{-1} = -A_1$$

~~Let $n \geq 0$~~

$$A_n = -A_{n-3}\, q^{n-3}$$

$$A_{3n} = -A_{3(n-1)}\, q^{3(n-1)}$$

$$= (-)(-A_{3(n-2)})\, q^{3(n-1)+3(n-2)}$$

$$= (-1)^n\, q^{3((n-1)+(n-2)+\cdots+2+1+0)}\, A_0$$

$$= (-1)^n\, q^{3n(n-1)/2}\, A_0 \quad \text{for } n \geq 0.$$

$$A_{-3n} = q^{+3n} A_{3n} = (-1)^n\, q^{3n(n+1)/2} A_0$$

Hence $A_{3n} = (-1)^n\, q^{3n(n-1)/2} A_0$ for all n.

$$A_{3n+1} = -A_{3(n-1)+1}\, q^{3(n-1)+1}$$

$$= (-1)^n A_1\, q^{(3(n-1)+1)+(3(n-2)+1)+\cdots(3\cdot 0+1)}$$

$$= (-1)^n A_1\, q^{3n(n-1)/2+n} = (-1)^n A_1\, q^{n(3n-1)/2}$$

for $n \geq 0$.

~~A_{3n+1}~~

$$A_{3n+2} = -A_{3(n-1)+2}\, q^{3(n-1)+2}$$

$$= (-1)^n A_2\, q^{3(n-1)+2+3(n-2)+2+\cdots 3(0)+2}$$

$$A_{3n+2} = (-1)^{n+1} A_1\, q_0^{3n(n-1)/2+2n} \qquad (18)$$

$$= (-1)^{n+1} A_1\, q_0^{n(3n+1)/2} \quad \text{for } n \geq 0.$$

Let $n>0$,

$$A_{-3n+1} = A_{-(3n-1)} = q_0^{3n-1} A_{3n-1}$$

$$= q_0^{3n-1} A_{3(n-1)+2} = q_0^{3n-1} (-1)^n A_1\, q_0^{(n-1)(3n-2)/2}$$

$$= (-1)^n A_1\, q_0^{n(3n+1)/2}$$

Hence $A_{3n+1} = (-1)^n A_1\, q_0^{n(3n-1)/2}$ for all n.

Let $n>0$,

$$A_{-3n+2} = A_{-(3n-2)} = q_0^{3n-2} A_{3n-2}$$

$$= q_0^{3n-2} A_{3(n-1)+1} = q_0^{3n-2} A_1\, q_0^{(n-1)(3n-4)/2} (-1)^{n-1}$$

$$= (-1)^{n-1} A_1\, q_0^{n(3n-1)/2}, \quad \&$$

$$A_{3n+2} = (-1)^{n+1} A_1\, q_0^{n(3n+1)/2} \quad \text{for all } n.$$

Therefore

$$F(a,b,q_0) = \sum_n A_{3n}\, a^{3n} + \sum_n A_{3n+1}\, a^{3n+1} + \sum_n A_{3n+2}\, a^{3n+2}$$

$$= \sum_n A_0 (-1)^n q_0^{3n(n-1)/2} a^{3n} + \sum_n (-1)^n A_1\, q_0^{n(3n-1)/2}\, a^{3n+1}$$

$$+ \sum_n (-1)^{n+1} A_1\, q_0^{n(3n+1)/2}\, a^{3n+2}$$

So

$$F(a,b,q) = A_0(b,q) \sum_n (-1)^n q^{3n(n-1)/2} a^{3n} \tag{19}$$

$$+ A_1(b,q)\Big(\sum_n (-1)^n q^{n(3n-1)/2} a^{3n+1}$$

$$+ \sum_n (-1)^{n+1} q^{n(3n+1)/2} a^{3n+2} \Big)$$

$$F(q, b, q^3) = (q;q^3)_\infty (q^2;q^3)_\infty \; (b;q^3)_\infty (q^3/b;q^3)_\infty$$

$$(q/b;q^3)_\infty (q^2 b;q^3)_\infty \; (qb;q^3)_\infty \; (q^2/b;q^3)_\infty$$

$$(q^3;q^3)_\infty^2$$

$$= (q)_\infty \, (b)_\infty (q/b)_\infty \quad (q^3;q^3)_\infty$$

$$= A_0(b,q^3) \sum_n (-1)^n q^{3n(3n-1)/2}$$

$$+ A_1(b,q^3)\Big(q \sum_n (-1)^n q^{3n(3n+1)/2}$$

$$+ q^2 \sum_n (-1)^{n+1} q^{\frac{9}{2}n(n+1)} \Big)$$

Note By JTP,

$$(z)_\infty (q/z)_\infty (q)_\infty = \sum_n (-1)^n z^n q^{n(n-1)/2}$$

Hence

$$(q)_\infty^2 (1)_\infty = 0 = \sum_n (-1)^n q^{n(n+1)/2}$$

&

$$\sum_n (-1)^{n+1} q^{\frac{9}{2}n(n+1)} = 0.$$

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Also,

$$(q^3;q^3)_\infty = \sum_n (-1)^n q^{3n(3n-1)/2} = \sum_n (-1)^n q^{3n(3n+1)/2}$$

and

$$(b)_\infty (q/b)_\infty (q)_\infty = A_0(b,q^3) + q A_1(b,q^3)$$

$$\sum_n (-1)^n b^n q^{n(n-1)/2} = \underbrace{A_0(b,q^3) + q A_1(b,q^3)}_{\text{3-dissection (in } q)}$$

n	$\frac{n(n-1)}{2}$ (mod 3)
0	0
1	0
2	1

Hence,

$A_0(b,q^3)$ contains only terms of the form q^{3m} in its q-expan

so

$$A_0(b,q^3) = \sum_{n \equiv 0,1 \pmod 3} (-1)^n b^n q^{n(n-1)/2}$$

$$= \sum_j (-1)^{3j} b^{3j} q^{3j(3j-1)/2} + \sum_j (-1)^{3j+1} b^{3j+1} q^{(3j+1)(3j)/2}$$

$$= \sum_j (-1)^j b^{-3j} q^{3j(j+1)/2} - \sum_j (-1)^j b^{3j+1} q^{3j(3j+1)/2}$$

Hence $A_0(b,q^3) = \sum_j (-1)^j (b^{-3j} - b^{3j+1}) q^{j(3j+1)/2}$.

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$$q\,A_1(b,q^3) = \sum_{n\equiv 2 \pmod 3} (-1)^n b^{n} q^{n(n-1)/2}$$

$$= \sum_i (-1)^{3i+2} b^{3i+2} q^{(3i+2)(3i+1)/2}$$

$$= q \sum_i (-1)^i b^{3i+2} q^{9i(i+1)/2},$$

and

$$A_1(b,q^3) = \sum_i (-1)^i b^{3i+2} q^{3i(i+1)/2}$$

$$= \sum_{i=0}^{\infty} (-1)^i b^{3i+2} q^{3i(i+1)/2} + \sum_{i=-1}^{-\infty} (-1)^i b^{3i+2} q^{3i(i+1)/2}$$

$$(i \to -1-i)$$

$$= \sum_{i=0}^{\infty} (-1)^i b^{3i+2} q^{3i(i+1)/2} + \sum_{i=0}^{\infty} (-1)^{i+1} b^{-3i-1} q^{3i(i+1)/2}$$

$$= \sum_{i=0}^{\infty} (-1)^i \left(b^{3i+2} - b^{-3i-1}\right) q^{3i(i+1)/2}$$

Hence

$$F(a,b,q) = \sum_j (-1)^j \left(b^{-3j} - b^{3j+1}\right) q^{j(3j+1)/2} \cdot \sum_{i=0}^{\infty} (-1)^i \left(a^{-3i} - a^{3i+3}\right) q^{3i(i+1)/2}$$

$$+ \sum_{i=0}^{\infty} (-1)^i \left(b^{3i+2} - b^{-3i-1}\right) q^{3i(i+1)/2} \cdot \sum_j (-1)^j \left(a^{-3j+1} - a^{3j+2}\right) q^{j(3j+1)/2}. \quad \square$$

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$(b)_\infty = (1-b)(bq)_\infty$. We divide both sides of Winquist's identity by $1-b$ & let $b \to 1$.

$$\frac{b^{-3j} - b^{3j+1}}{1-b} = \frac{b^{-3j}(1-b^{6j+1})}{1-b} \longrightarrow 6j+1$$

$$\frac{b^{3i+2} - b^{-3i-1}}{1-b} = \frac{b^{-3i-1}(b^{6i+3}-1)}{1-b} \longrightarrow -(6i+3) = -3(2i+1)$$

We obtain

$$(a)_\infty^3 (q/a)_\infty^3 (q)_\infty^4$$

$$= \sum_{i=0}^{\infty}\sum_{j=-\infty}^{\infty} (-1)^{i+j}\Big((6j+1)(a^{-3i} - a^{3i+3})$$
$$-3(2i+1)(a^{-3j+1} - a^{3j+2})\Big)$$
$$\cdot q^{\frac{3}{2}i(i+1)+j(3j+1)/2}.$$

Next, $(a)_\infty^3 = (1-a)^3 (aq)_\infty^3$, ~~we divide both sides by [illegible] & let a→1~~.

$LHS = (1-a)^3\left((q)_\infty^{10} + c_1(a-1) + c_2(a-1)^2 + \cdots\right)$

(since analytic near $a=1$)

$= -(q)_\infty^{10}(a-1)^3 + c_1(a-1)^4 + \cdots$

Let $R(a) = RHS$ as a function of a near $a=1$.

Then $R(1) = R'(1) = R''(1) = 0$.

$$\text{Coeff of } (a-1)^3 = \frac{R'''(a)}{3!}$$

$$\left[\left(\frac{d}{da}\right)^3 \left(a^{-3i} - a^{3i+3}\right)\right]_{a=1} = (-3i)(-3i-1)(-3i-2) - (3i+3)(3i+2)(3i+1) \tag{23}$$

$$= -3(3i+2)(3i+1)(2i+1)$$

$$\left[\left(\frac{d}{da}\right)^3 \left(a^{-3j+1} - a^{3j+2}\right)\right]_{a=1} = (-3j+1)(-3j)(-3j-1)$$

$$- (3j+2)(3j+1)(3j)$$

$$= -3j(3j+1)(6j+1).$$

Hence,

$$(q)_\infty^{10} = \sum_{i=0}^{\infty}\sum_{j=-\infty}^{\infty} \frac{(-1)^{i+j}}{2}\Big((6j+1)(2i+1)(3i+2)(3i+1)$$

$$- (2i+1)(3j)(3j+1)(6j+1)\Big)$$

$$\cdot\, q^{\frac{3}{2}i(i+1)+j(3j+1)/2}$$

$$= \sum_{i=0}^{\infty}\sum_{j=-\infty}^{\infty} (-1)^{i+j}\big((2i+1)(6j+1)\big)$$

$$\left(\frac{(3i+1)(3i+2)}{2} - \frac{(3j)(3j+1)}{2}\right)$$

$$\cdot\, q^{\frac{3}{2}i(i+1)+j(3j+1)/2}.$$

Theorem (Ramanujan)

$$p(11n+6) \equiv 0 \pmod{11} \qquad \text{for } n \geq 0.$$

Proof.

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q)_\infty} = \frac{(q)_\infty^{10}}{(q)_\infty^{11}}$$

$$\equiv \frac{(q)_\infty^{10}}{(q^{11};q^{11})_\infty} \pmod{11}$$

Hence

$$(*) \qquad \sum_{n=0}^{\infty} p(11n+6) q^{11n+6} \equiv \frac{1}{(q^{11};q^{11})_\infty} \sum_{n=0}^{\infty} P_{10}(11n+6) q^{11n+6}$$

where

$$\sum_{n=0}^{\infty} P_{10}(n) q^n = (q)_\infty^{10}$$

$$= \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} (-1)^{i+j} (2i+1)(6j+1) \left[\frac{(3i+1)(3i+2)}{2} - \frac{3j(3j+1)}{2} \right] q^{3i(i+1)/2 + j(3j+1)/2}$$

Suppose $3i(i+1)/2 + j(3j+1)/2 \equiv 6 \pmod{11}$

$\Leftrightarrow \quad 3(i^2+i) + 3j^2+j \equiv 12 \equiv 1 \pmod{11}$

$\Leftrightarrow \quad 3(i^2+12i) + 3(j^2+4j) \equiv 1 \pmod{11}$

$\Leftrightarrow \quad i^2+12i + j^2+4j \equiv 4 \pmod{11}$

$\Leftrightarrow \quad (i^2+12i+36) + (j^2+4j+4) \equiv 0 \pmod{11}$

$\Leftrightarrow \quad (i+6)^2 + (j+2)^2 \equiv 0 \pmod{11}$

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x	$x^2 \pmod{11}$
0	0
± 1	1
± 2	4
± 3	9
± 4	5
± 5	$25 \equiv 3$

So $x^2 \equiv 0, 1, 3, 4, 5, 9 \pmod{11}$

$$(i+6)^2 + (j+2)^2 \equiv \begin{matrix} 0, & 1, & 3, & 4, & 5, & 9 \\ 1, & 2, & 4, & 5, & 6, & 10 \\ 3, & 4, & 6, & 7, & 8, & 1 \\ 4, & 5, & 7, & 8, & 9, & 2 \\ 5, & 6, & 8, & 9, & 10, & 3 \\ 9, & 10, & 1, & 2, & 3, & 7 \end{matrix} \pmod{11}$$

$(i+6)^2 + (j+2)^2 \equiv 0 \pmod{11}$

iff $i+6 \equiv 0$ & $j+2 \equiv 0 \pmod{11}$

ie $i \equiv 5$ & $j \equiv 9 \pmod{11}$

in which case $(2i+1) \equiv 10+1 \equiv 0 \pmod{11}$,

& $(6j+1) \equiv 55 \equiv 0 \pmod{11}$.

So that $P_{10}(11n+6) \equiv 0 \pmod{11^2}$.

(*) then implies $p(11n+6) \equiv 0 \pmod{11}$. $\square$

The Combinatorics of Ramanujan's Partition Congruences

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DYSON'S RANK

Defn: Dyson (1944) defined the rank of a partition as the largest part minus the number of parts.

Example $\lambda = (7,7,6,5,3,3,1,1,1)$

$\text{rank}(\lambda) = 7 - 9 = -2.$

Defn: Let $N(m,n)$ denote the number of partitions of n with rank m.

Theorem: $N(m,n) = N(-m,n)$ for all m, n.

Proof: Let $\Omega(m,n)$ = Set of partitions of n with rank m.

Let λ be a partition.

largest part of λ' = # of parts of λ

& number of parts of λ' = largest part of λ

Hence $\text{rank}(\lambda') = -\text{rank}(\lambda)$ and the map $\lambda \mapsto \lambda'$ gives a bijection $\Omega(m,n) \to \Omega(-m,n)$.

Hence $N(m,n) = |\Omega(m,n)| = |\Omega(-m,n)| = N(-m,n)$. $\square$

NOTE: $N(m,n) = 0$ if $n \le |m|$.

Defn: Let $N(m,t,n)$ = # of partitions of n with rank $\equiv m \pmod{t}$. Let $t \geq 2$.

Theorem: $N(k,t,n) = N(t-k,t,n)$ for all k, n.

Proof: Let $t \geq 2$, $n \geq 1$.

$$N(k,t,n) = \sum_a N(at+k, n)$$

(where we note that this sum has only finitely many nonzero terms.)

$$= \sum_a N(-at-k, n)$$

$$= \sum_b N(-(1-b)t-k, n)$$

(since as b runs thru $\mathbb{Z}$ so does $a = 1-b$ and vice-versa)

$$= \sum_b N(bt+(t-k), n)$$

$$= N(t-k, t, n). \quad \square$$

<u>Dyson's Conjectures (1944)</u>

(1) $\quad N(m, 5, 5n+4) = \frac{1}{5}\, p(5n+4), \quad 0 \le m \le 4.$

(2) $\quad N(m, 7, 7n+5) = \frac{1}{7}\, p(7n+5), \quad 0 \le m \le 6$

<u>Note:</u>

(1) means that the residue of the rank mod 5 divides the partitions of $5n+4$ into five equal classes.

(2) . . . . mod 7 . . . . .
. . . . . $7n+5$ . . . seven . . . . . .

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__Example of (1) (n=0)__

Partition of 4	Rank	Rank (mod 5)
4	$4-1=3$	3
$3+1$	$3-2=1$	1
$2+2$	$2-2=0$	0
$2+1+1$	$2-3=-1$	4
$1+1+1+1$	$1-4=-3$	2

$$N(0,5,4) = N(1,5,4) = N(2,5,4) = N(3,5,4)$$
$$= N(4,5,4) = 1$$
$$N(0,5,4) + \cdots + N(4,5,4) = p(4) = 5\,N(0,5,4)$$
$$\text{So}\quad N(m,5,4) = p(4)/5.$$

The Dyson Conjectures were proved by Atkin & Swinnerton-Dyer in 1954. Their proof is analytic and depends on elliptic function and q-series identities. No combinatorial proof is known.

__Note__: ~~The analog of the Dyson conjecture for~~ The analog of (1)-(2) does not hold for partitions of $11n+6$.

__Example__

Partitions of 6	Rank	Rank (mod 11)
6	$6-1=5$	5
$5+1$	$5-2=3$	3
$4+2$	$4-2=2$	2
$4+1+1$	$4-3=1$	1

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$3+3$	$3-2=1$	1
$3+2+1$	$3-3=0$	0
$3+1+1+1$	$3-4=-1$	10
$2+2+2$	$2-3=-1$	10
$2+2+1+1$	$2-4=-2$	9
$2+1+1+1+1$	$2-5=-3$	8
$1+1+1+1+1+1$	$1-6=-5$	6

Hence $N(0,11,6)=1$

$N(1,11,6)=2$

$N(2,11,6)=1$

$N(3,11,6)=1$

$N(4,11,6)=0$

$N(5,11,6)=1$

$N(6,11,6)=1$

$N(7,11,6)=0$

$N(8,11,6)=1$

$N(9,11,6)=1$

$N(10,11,6)=2$

Dyson (1944) conjectured that exist some statistic he called the "crank" which would be like the rank and explain Ramanujan's congruence $p(11n+6) \equiv 0 \pmod{11}$.

The Andrews-Garvan Crank

For a partition λ let $\ell(\lambda)$ = the largest part, $\omega(\lambda)$ = # of ones in λ and $\mu(\lambda)$ = # of parts of $\lambda > \omega(\lambda)$. Then

$$\operatorname{crank}(\lambda) = \begin{cases} \ell(\lambda) & \text{if } \omega(\lambda)=0 \\ \mu(\lambda)-\omega(\lambda) & \text{if } \omega(\lambda)>0 \end{cases}$$

Let

$M(m,n)$ = # of partitions of n with crank m

and

$M(k,t,n)$ = # of partitions of n with crank $\equiv k \bmod t$.

Theorem (Andrews - G.)

(1) $M(-m,n) = M(m,n)$ for $n \geq 2$.

(2) $M(k,5,5n+4) = \dfrac{p(5n+4)}{5}$, $0 \leq k \leq 4$;

(3) $M(k,7,7n+5) = \dfrac{p(7n+5)}{7}$, $0 \leq k \leq 6$;

(4) $M(k,11,11n+6) = \dfrac{p(11n+6)}{11}$, $0 \leq k \leq 10$.

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<u>Example</u>

Partitions of 6	crank	(mod 11)
6	6	$\equiv 6$
5+1	$1-1=0$	$\equiv 0$
4+2	4	$\equiv 4$
4+1+1	$1-2=-1$	$\equiv 10$
3+3	3	$\equiv 3$
3+2+1	$2-1=1$	$\equiv 1$
3+1+1+1	$0-3=-3$	$\equiv 8$
2+2+2	2	$\equiv 2$
2+2+1+1	$0-2=-2$	$\equiv 9$
2+1+1+1+1	$0-4=-4$	$\equiv 7$
1+1+1+1+1+1	$0-6=-6$	$\equiv 5$

We see that

$$M(0,11,6)=M(1,11,6)=\cdots=M(10,11,6)=1=\frac{p(6)}{11}.$$

<u>Def</u>: An <u>algebraic number</u> α is an $\alpha\in\mathbb{C}$ that satisfies

$$a_n\alpha^n+a_{n-1}\alpha^{n-1}+\cdots+a_1\alpha+a_0=0$$

some polynomial of degree n where $a_j\in\mathbb{Z}$ & $a_n\neq 0$.

<u>Example</u> (1) $\sqrt{2}$ is algebraic since it satisfies $x^2-2=0$.

(2) $\zeta=e^{2\pi i/5}=\cos(\pi/5)+i\sin(\pi/5)$ is <u>algebraic</u> since it satisfies $z^5-1=0$.

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The *minimal polynomial* of an algebraic number α is the unique (irreducible) monic polynomial of smallest degree $p(x)$ with rational coefficients such that
$$p(\alpha)=0.$$

Example: The minimal polynomial of $\sqrt{2}$ is x^2-2.

Theorem Let p be prime. The minimal polynomial of $\zeta_p = \exp(2\pi i/p) = \cos\left(\frac{2\pi}{p}\right) + i\sin\left(\frac{2\pi}{p}\right)$ is $p(x) = 1+x+x^2+\cdots+x^{p-1}$.

Definition: A *vector partition* π is a triple
$$\pi = (\pi_1, \pi_2, \pi_3)$$
where π_1 is a partition into distinct parts, and π_2, π_3 ~~is~~ are partitions ~~into~~ (unrestricted).
We say π is a vector partition of n if
$$n = |\pi_1| + |\pi_2| + |\pi_3|.$$

Example $\pi = ((5,3,2), (2,2), (5,1,1))$ is a ~~p~~ vector partition of 21.

Definition: ~~We define~~ For a vector partition $\pi = (\pi_1, \pi_2, \pi_3)$ we define a weight $W(\pi)$ and crank(π) by
$$W(\pi) = (-1)^{\#(\pi_1)}$$
and
$$\text{crank}(\pi) = \#(\pi_2) - \#(\pi_3).$$

Let V = set of vector partitions $= \{(\pi_1, \pi_2, \pi_3) : \pi_1$ is a partition into distinct parts & π_2, π_3 are partitions$\}$.

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__Defn__ Let $N_V(m,n)$ = # of vector partitions of n with crank m counted according to the weight w, ie

$$N_V(m,n) = \sum_{\substack{\pi \in V \\ |\pi| = n \\ \mathrm{crank}(\pi) = m}} W(\pi)$$

```
[[[], [], [1, 1]]      1    -2]
[                              ]
[ [[], [], [2]]        1    -1]
[                              ]
[ [[], [1], [1]]       1     0]
[                              ]
[[[], [1, 1], []]      1     2]
[                              ]
[ [[], [2], []]        1     1]
[                              ]
[ [[1], [], [1]]      -1    -1]
[                              ]
[ [[1], [1], []]      -1     1]
[                              ]
[ [[2], [], []]       -1     0]
```

There are eight vector partitions of $n=2$.

$N_V(2,2) = 1$

$N_V(1,2) = 1 - 1 = 0$

$N_V(0,2) = 1 - 1 = 0$

$N_V(-1,2) = 1 - 1 = 0$

$N_V(-2,2) = 1$

__Theorem__ Let $|q|<1$ & $|q| < |z| < \frac{1}{|q|}$.

$$\sum_{n=0}^{\infty} \sum_{m} N_V(m,n)\, z^m q^n = \frac{(q)_\infty}{(zq)_\infty (z^{-1}q)_\infty} = \prod_{n=1}^{\infty} \frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)}$$

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Proof

$$\prod_{n=1}^{\infty}\frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)} = \prod_{n=1}^{\infty}(1-q^n)\cdot\prod_{n=1}^{\infty}\frac{1}{1-zq^n}\cdot\prod_{n=1}^{\infty}\frac{1}{1-z^{-1}q^n}$$

$$= \sum_{\pi_1\in\mathcal{PD}}(-1)^{\#(\pi_1)}q^{|\pi_1|}\sum_{\pi_2\in\mathcal{P}}z^{\#(\pi_2)}q^{|\pi_2|}\sum_{\pi_3\in\mathcal{P}}z^{-\#(\pi_3)}q^{|\pi_3|}$$

$$= \sum_{\pi=(\pi_1,\pi_2,\pi_3)\in V=\mathcal{PD}\times\mathcal{P}\times\mathcal{P}}(-1)^{\#(\pi_1)}z^{\#(\pi_2)-\#(\pi_3)}q^{|\pi_1|+|\pi_2|+|\pi_3|}$$

$$= \sum_{\pi\in V}w(\pi)\,z^{\mathrm{crank}(\pi)}q^{|\pi|}$$

$$= \sum_{n\geq 0}\sum_{m}\Bigg(\sum_{\substack{\pi\in V\\ \mathrm{crank}(\pi)=m\\ |\pi|=n}}w(\pi)\Bigg)z^m q^n$$

$$= \sum_{n\geq 0}\sum_{m}N_V(m,n)\,z^m q^n. \qquad \square$$

Corollary

(1) $$N_V(m,n) = N_V(-m,n)$$

for all m, n

(2) $$N_V(k,t,n) = N_V(t-k,t,n)$$

for all k, n, t, where $t \geq 1$,

where

$$N_V(k,t,n) = \sum_{\substack{|\pi|=n\\ \mathrm{crank}(\pi)\equiv k \pmod t}} w(\pi).$$

(3) $$p(n) = \sum_m N_V(m,n). \qquad (35)$$

Proof (1) Let $F(z) = \dfrac{(q)_\infty}{(zq)_\infty(z^{-1}q)_\infty}$.

Then $F(\frac{1}{z}) = F(z)$ so

$$\sum_n \sum_m N_V(m,n) z^{-m} q^n = \sum_n \sum_m N_V(m,n) z^m q^n$$

$$\sum_n \sum_m N_V(-m,n) z^m q^n = \sum_n \sum_m N_V(m,n) z^m q^n,$$

and

$$N_V(-m,n) = N_V(m,n) \quad \text{for all } m,n. \blacksquare$$

(2) as before. (3) follows by letting $z=1$. $\square$

Theorem (G, 1986)

(1) $N_V(k,5,5n+4) = \dfrac{p(5n+4)}{5}, \quad 0 \le k \le 4,$

(2) $N_V(k,7,7n+5) = \dfrac{p(7n+5)}{7}, \quad 0 \le k \le 6,$

(3) $N_V(k,11,11n+6) = \dfrac{p(11n+6)}{11}, \quad 0 \le k \le 10.$

See next page for an example of (1).

(40)

Proof of (1)

We need $(q)_\infty = \sum_n (-1)^n q^{n(3n-1)/2}$

and the following form of JTP:

$$(zq)_\infty (z^{-1}q)_\infty (q)_\infty = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} z^{-n} \frac{(z^{2n+1}-1)}{(z-1)}$$

for $|q|<1$, $z \neq 0,1$. (See p.23 of Ch2 NOTES).

~~Reco~~

Let $\zeta = e^{2\pi i/5}$ so that $\zeta^5 = 1$ & $1+\zeta+\zeta^2+\zeta^3+\zeta^4 = 0$.

Recall that

$$1-z^5 = (1-z)(1-\zeta z)(1-\zeta^2 z)(1-\zeta^3 z)(1-\zeta^4 z).$$

So that

$$(q)_\infty (\zeta q)_\infty (\zeta^2 q)_\infty (\zeta^3 q)_\infty (\zeta^4 q)_\infty$$

$$= \prod_{n=1}^{\infty} \prod_{k=0}^{4} (1-\zeta^k q^n) = \prod_{n=1}^{\infty}(1-q^{5n}) = (q^5;q^5)_\infty.$$

Next we let $z=\zeta$ in

$$\sum_{n\geq 0} \sum_m N_V(m,n) z^m q^n = \frac{(q)_\infty}{(zq)_\infty (q/z)_\infty}$$

to obtain

$$(*) \quad \sum_{n\geq 0} \sum_m N_V(m,n) \zeta^m q^n = \frac{(q)_\infty}{(\zeta q)_\infty (q/\zeta)_\infty}$$

Now,

$$\sum_{n\geq 0}\sum_{m} N_V(m,n)\zeta^m q^n$$

$$=\sum_{n\geq 0}\left(\sum_{k=0}^{4}\sum_{m\equiv k\,(\mathrm{mod}\ 5)} N_V(m,n)\,\zeta^m\right) q^n$$

$$=\sum_{n\geq 0}\left(\sum_{k=0}^{4}\sum_{m\equiv k\,(\mathrm{mod}\ 5)} N_V(m,n)\,\zeta^k\right) q^n$$

$$=\sum_{n\geq 0}\left(\sum_{k=0}^{4}\left(\sum_{m\equiv k\,(\mathrm{mod}\ 5)} N_V(m,n)\right)\zeta^k\right) q^n$$

$$=\sum_{n\geq 0}\left(\sum_{k=0}^{4} N_V(k,5,n)\,\zeta^k\right) q^n$$

Therefore,

$$(*)\quad \sum_{n\geq 0}\left(\sum_{k=0}^{4} N_V(k,5,n)\,\zeta^k\right) q^n = \frac{(q)_\infty}{(\zeta q)_\infty(\zeta^4 q)_\infty} \qquad \text{since } \zeta^5=1\ \&\ \zeta^{-1}=\zeta^4$$

$$= \frac{(q)_\infty\,(q)_\infty\,(\zeta^2 q)_\infty\,(\zeta^3 q)_\infty}{(q)_\infty\,(\zeta q)_\infty\,(\zeta^2 q)_\infty\,(\zeta^3 q)_\infty\,(\zeta^4 q)_\infty}$$

$$= \frac{(q)_\infty\,(\zeta^2 q)_\infty\,(\zeta^{-2} q)_\infty\,(q)_\infty}{(q^5;q^5)_\infty} \qquad \text{since } \zeta^5=1\ \&\ \zeta^{-2}=\zeta^3$$

(42)

$$= \frac{\sum_{i} (-1)^i q^{i(3i-1)/2} \sum_{j\ge 0} (-1)^j q^{j(j+1)/2} \zeta^{-2j} \frac{(1-\zeta^{2(2j+1)})}{(1-\zeta^2)}}{(q^5;q^5)_\infty}$$

$$= \frac{\sum_{\substack{i\in\mathbb{Z}\\ j\ge 0}} (-1)^{i+j} \zeta^{-2j} \frac{(1-\zeta^{2(2j+1)})}{(1-\zeta^2)} q^{i(3i-1)/2 + j(j+1)/2}}{(q^5;q^5)_\infty}$$

We show that the coeff. of $q^{5n+4} = 0$.

$$\frac{i(3i-1)}{2} + \frac{j(j+1)}{2} \equiv 4 \pmod 5$$

$\iff 3i^2 - i + j^2 + j \equiv 3 \pmod 5$

$\iff 3(i^2-2i) + j^2 - 4j \equiv 3 \pmod 5$

$\iff 3(i^2-2i+1) + (j^2-4j+4) \equiv 0 \pmod 5$

$\iff 3(i-1)^2 + (j-2)^2 \equiv 0 \pmod 5$

$\iff i \equiv 1 \pmod 5$ & $j \equiv 2 \pmod 5$ $(\underline{EX})$

In which case $2(2j+1) \equiv 2(5) \equiv 0 \pmod 5$

and $1 - \zeta^{2(2j+1)} = 0$.

It follows that the coeff of q^{5n+4} on RHS$(*) = 0$.

Hence
$$\sum_{k=0}^{4} N_V(k, 5, 5n+4)\, \zeta^k = 0,$$

and ζ is a root of the polynomial

$$p(x) = \sum_{k=0}^{4} N_V(k, 5, 5n+4)\, x^k \in \mathbb{Z}[x].$$

(43)

But the minimal polynomial of ζ (over $\mathbb{Q}$) is
$$p_{\zeta}(x) = \sum_{k=0}^{4} x^k = 1+x+x^2+x^3+x^4.$$

It follows that
$$N_V(0,5,5n+4) = N_V(1,5,5n+4) = N_V(2,5,5n+4)$$
$$= N_V(3,5,5n+4) = N_V(4,5,5n+4).$$
But
$$\sum_{k=0}^{4} N_V(k,5,5n+4) = p(5n+4)$$
so that
$$5N_V(0,5,5n+4) = p(5n+4)$$
&
$$N_V(k,5,5n+4) = \frac{p(5n+4)}{5} \quad \text{for } 0\le k\le 4. \ \square$$

Note: The proofs of (2), (3) are similar except that (3) requires Winquist's Identity.

Theorem (Andrews & G.)
$$M(m,n) = N_V(m,n)$$
for all $n \geq 2$ & for all m.

Proof: We need the q-binomial thm:

$$(*) \qquad \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(q)_n} = \frac{(az)_\infty}{(z)_\infty} \qquad (\text{for } |q|,|z|<1).$$

Suppose $|q|<1$ & $|q|<|z|<1/|q|$.

(44)

$$\sum_{n=0}^{\infty} \sum_{m} N_V(m,n) z^m q^n = \frac{(q)_\infty}{(zq)_\infty (q/z)_\infty}$$

$$= \frac{(1-q)}{(zq)_\infty} \frac{(q^2;q)_\infty}{(q/z)_\infty}$$

$$= \frac{(1-q)}{(zq)_\infty} \sum_{j=0}^{\infty} \frac{(zq)_j (q/z)^j}{(q)_j} \qquad \left(z \to q/z \text{ in } (*) \quad a \to qz \right)$$

(since $|q| < 1$ & $|\frac{q}{z}| < 1$)

$$= \frac{(1-q)}{(zq)_\infty} \left(1 + \sum_{j=1}^{\infty} \frac{(zq)_j (q/z)^j}{(q)_j} \right)$$

$$= \frac{(1-q)}{(zq)_\infty} \left(1 + \sum_{j=1}^{\infty} \frac{(zq)_j (q/z)^j}{(1-q)(q^2;q)_{j-1}} \right)$$

$$= \frac{(1-q)}{(zq)_\infty} + \sum_{j=1}^{\infty} \frac{q^j z^{-j}}{(q^2;q)_{j-1} (zq^{j+1})_\infty}$$

$$= \frac{(1-q)}{(zq)_\infty} + \underbrace{\sum_{j=1} \frac{z^{-j} q^{\overbrace{1+1+\cdots+1}^{j\text{-times}}}}{(1-q^2)(1-q^3)\cdots(1-q^j)(1-zq^{j+1})(1-zq^{j+2})\cdots}}$$

G.F. for ptns with at least one 1
in which the power of z counts
# of parts large than the number of ones — # of ones

(45)

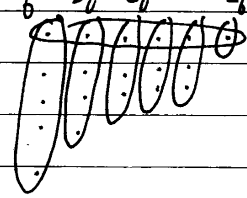
$$\frac{z^{-j}q^{j}}{(q^2;q)_{j-1}(zq^{j+1})_\infty} = \sum_{\substack{\pi\in P\\ \omega(\pi)=j}} z^{\mathrm{crank}(\pi)} q^{|\pi|}$$

and

$$\sum_{j=1}^{\infty} \frac{z^{-j}q^{j}}{(q^2;q)_{j-1}(zq^{j+1})_\infty} = \sum_{\substack{\pi\in P\\ \omega(\pi)\geq 1}} z^{\mathrm{crank}(\pi)} q^{|\pi|}$$

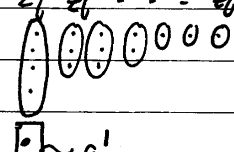
$$\frac{1}{(zq)_\infty} = \sum_{\pi\in P} z^{\ell(\pi)} q^{|\pi|}$$

$$zq^{n_1}\, zq^{n_2}\, zq^{n_3}\cdots zq^{n_k} = z^{k} q^{n_1+n_2+\cdots+n_k}$$



$$\frac{q}{(zq)_\infty} = \sum_{\pi\in P_1} z^{\ell'(\pi)} q^{|\pi|}$$

$zq^{n_1}\, zq^{n_2} \cdots zq^{n_k}$



$\sim q^1$

where P_1 = set of partitions with at least one 1 &

$$\ell'(\pi) = \begin{cases} \ell(\pi) & \text{if } \pi\neq(1) \\ 0 & \text{if } \pi=(1) \end{cases}$$

Hence,

$$\frac{(1-q)}{(zq)_\infty} = 1 + (z-1)q^1 + \sum_{\substack{\pi\in P\\ |\pi|\geq 2}} z^{\ell(\pi)} q^{|\pi|} - \sum_{\substack{\pi\in P_1\\ |\pi|\geq 2}} z^{\ell(\pi)} q^{|\pi|}$$

$$= 1 + (z-1)q^1 + \sum_{\substack{\pi\in P\setminus P_1\\ |\pi|\geq 2}} z^{\ell(\pi)} q^{|\pi|}$$

(C46)

$$= 1 + (z-1)q^1 + \sum_{\substack{\omega(\pi)=0 \\ \pi\in P \\ |\pi|\geq 2}} z^{\mathrm{crank}(\pi)} q^{|\pi|}.$$

Therefore,

$$\sum_{n=0}^{\infty} \sum_{m} N_V(m,n) z^m q^n = \frac{(q)_\infty}{(zq)_\infty (z^{-1}q)_\infty}$$

$$= 1 + (-1+z+z^{-1})q + \sum_{\substack{\pi\in P \\ |\pi|\geq 2}} z^{\mathrm{crank}(\pi)} q^{|\pi|}$$

$$= 1 + (-1+z+z^{-1})q + \sum_{n=2}^{\infty} \sum_{m} M(m,n) z^m q^n$$

and

$$N_V(m,n) = M(m,n)$$

for $n \geq 2$. $\Box$

Corollary (Andrews 86.)

(1) $M(-m,n) = M(m,n)$ for $n \geq 2$.

(2) $M(k,5,5n+4) = \dfrac{p(5n+4)}{5}$ for $0 \leq k \leq 4$.

(3) $M(k,7,7n+5) = \dfrac{p(7n+5)}{7}$ for $0 \leq k \leq 6$.

(4) $M(k,11,11n+6) = \dfrac{p(11n+6)}{11}$ for $0 \leq k \leq 11$.

Other Cranks

Let $t \geqslant 2$.

Defn Given the Ferrers diagram of a partition π we label node (i,j) (ith row & jth col.) by least nonneg residue of $j - i \pmod t$.
The resulting diagram is called a t-residue diagram.

Eg $\pi = (11, 7, 3, 3)$, $t=5$.

$$\begin{array}{ccccccccccc}
0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 \\
4 & 0 & 1 & 2 & 3 & 4 & 0 & & & & \\
3 & 4 & 0 & & & & & & & & \\
2 & 3 & 4 & & & & & & & &
\end{array}$$

For each $0 \le i \le t-1$ let

$$r_i = r_i(\pi) = \#\text{ of nodes labelled } i \pmod t \text{ in the } t\text{-residue diagram of } \pi.$$

For example, for π above

$$(r_0, r_1, r_2, r_3, r_4) = (6, 3, 4, 5, 6).$$

Define

$$t\text{-core-crank}(\pi) := \sum_{j=0}^{t-1} \left(j - \left(\tfrac{t-1}{2}\right)\right)^{t-3} (r_j - r_{j+1}),$$

where $r_t := r_0$.

Theorem: (G., Stanton & Kim, 1990)
Let $(t, \delta) = (5,4)$, $(7,5)$ or $(11,6)$.
Then the t-core-crank $\pmod t$ divides the partitions of $tn+\delta$ into t equal classes.

(48)

<u>Example</u> $t=5$, $5n+4=4$.

$$5\text{-core-crank}(\pi) = \sum_{j=0}^{4} (j-2)^2 (r_j - r_{j+1})$$

$$\equiv r_1 + 2r_2 + 3r_3 + 4r_4 \pmod{5}.$$

<u>Ptns of 4</u>		5-core-crank (mod 5)
4	$\begin{array}{cccc} 0 & 1 & 2 & 3 \end{array}$	$1+2+3 \equiv 1$
3+1	$\begin{array}{ccc} 0 & 1 & 2 \\ 4 & & \end{array}$	$1+2+4 \equiv 2$
2+2		
2+1+1	$\begin{array}{cc} 0 & 1 \\ 4 & 0 \end{array}$	$1+4 \equiv 0$
1+1+1+1		
	$\begin{array}{cc} 0 & 1 \\ 4 & \\ 3 & \end{array}$	3
	$\begin{array}{c} 0 \\ 4 \\ 3 \\ 2 \end{array}$	4