It follows that the power series expansion (11)

of $\dfrac{(q)_\infty^3}{(q^{25};q^{25})_\infty^3}$ does not have terms involving q^{5n+2}, q^{5n+4}.

Hence

$$J_0 J_2 + 1 = 0$$

and

$$J_0 J_2 = -1,$$

$$J_2 = -\frac{1}{J_0}.$$

Hence

$$\frac{(q)_\infty}{(q^{25};q^{25})_\infty} = J_0(q^5) - q - \frac{q^2}{J_0(q^5)}$$

and

$$\frac{1}{(q)_\infty} = (q^{25};q^{25})_\infty \times \frac{1}{J_0(q^5) - q - \dfrac{q^2}{J_0(q^5)}}$$

Let $\zeta = e^{2\pi i/5}$. Then

$$z^5 - 1 = (z-1)(z-\zeta)(z-\zeta^2)(z-\zeta^3)(z-\zeta^4)$$

$$= (-1)(1-z)(-\zeta)(1-\zeta^4 z)(-\zeta^2)(1-\zeta^3 z)$$

$$(-\zeta^3)(1-\zeta^2 z)(-\zeta^4)(1-\zeta z), \text{ since } \zeta^5 = 1$$

$$= (-1)^5 \zeta^{1+2+3+4} (1-z)(1-\zeta z)(1-\zeta^2 z)(1-\zeta^3 z)(1-\zeta^4 z)$$

$$z^5 - 1 = -\zeta^{10} (1-z)(1-\zeta z)(1-\zeta^2 z)(1-\zeta^3 z)(1-\zeta^4 z)$$

&

$$(1-z)(1-\zeta z)(1-\zeta^2 z)(1-\zeta^3 z)(1-\zeta^4 z) = 1 - z^5$$

$$\prod_{j=0}^{4} (1-\zeta^j z) = (1-z^5)$$