Claim: $\prod_{j=0}^{4}\left(1-(\zeta^k)^j z\right) = \begin{cases} (1-z^5) & 1\le k\le 4 \\ (1-z)^5 & k=0 \end{cases}$ (12)

~~for k = 1, 2, 3, 4.~~

(EX)

Let $E(q)=(q)_\infty = \prod_{n=0}^{\infty}(1-q^{5n+5})(1-q^{5n+1})(1-q^{5n+2})(1-q^{5n+3})(1-q^{5n+4})$

$$= \prod_{n=0}^{\infty}(1-q^{5n+5}) \prod_{k=1}^{4}\prod_{n=0}^{\infty}(1-q^{5n+k})$$

$$E(\zeta^l q) = \prod_{n=0}^{\infty}\left(1-(\zeta^l q)^{5n+5}\right) \prod_{n=0}^{\infty}\prod_{k=1}^{4}\left(1-(\zeta^l q)^{5n+k}\right)$$

$$= \prod_{n=0}^{\infty}(1-q^{5n+5}) \prod_{n=0}^{\infty}\prod_{k=1}^{4}\left(1-(\zeta^k)^l q^{5n+k}\right)$$

$E(q)E(\zeta q)E(\zeta^2 q)E(\zeta^3 q)E(\zeta^4 q)$

$$= \prod_{l=0}^{4} E(\zeta^l q)$$

$$= \prod_{n=0}^{\infty}(1-q^{5n+5})^5 \prod_{n=0}^{\infty}\prod_{k=1}^{4}\prod_{l=0}^{4}\left(1-q^{5n+k}(\zeta^k)^l\right)$$

$$= (q^5;q^5)_\infty^5 \prod_{n=0}^{\infty}\prod_{k=1}^{4}\left(1-q^{5(5n+k)}\right)$$

$$= (q^5;q^5)_\infty^5 (q^5;q^{25})_\infty (q^{10};q^{25})_\infty (q^{15};q^{25})_\infty (q^{20};q^{25})_\infty \times \frac{(q^{25};q^{25})_\infty}{(q^{25};q^{25})_\infty}$$

$$= \frac{(q^5;q^5)_\infty^6}{(q^{25};q^{25})_\infty} = \frac{E(q^5)^6}{E(q^{25})}$$