Winquist's Identity (1969)

Suppose $a, b \neq 0$ and $|q|<1$. Then

$$(a)_\infty (q/a)_\infty (b)_\infty (q/b)_\infty \left(\frac{a}{b}\right)_\infty \left(\frac{bq}{a}\right)_\infty$$
$$(ab)_\infty \left(\frac{q}{ab}\right)_\infty (q)_\infty^2$$
$$= \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} (-1)^{i+j} \Big((a^{-3i} - a^{3i+3})(b^{-3j} - b^{3j+1})$$
$$+ (a^{-3j+1} - a^{3j+2})(b^{3i+2} - b^{-3i-1}) \Big)$$
$$\cdot\, q^{\frac{3}{2} i(i+1) + j(3j+1)/2} .$$

Proof:

Let
$$F(a,b,q) := (a)_\infty (q/a)_\infty (b)_\infty (q/b)_\infty \left(\frac{a}{b}\right)_\infty \left(\frac{bq}{a}\right)_\infty$$
$$(ab)_\infty \left(\frac{q}{ab}\right)_\infty (q)_\infty^2 .$$

Let $f(a) = (a)_\infty (q/a)_\infty$
$$f(aq) = (aq)_\infty (1/a)_\infty = f\left(\frac{1}{a}\right)$$
$$= f(a) \cdot \frac{(1-a^{-1})}{(1-a)} = -a^{-1} \frac{(1-a)}{(1-a)} f(a)$$
$$= -a^{-1} f(a).$$

$$F(a,b,q) := f(a) f(b) f\left(\frac{a}{b}\right) f(ab) \quad (q)_\infty^2$$

$$F(aq,b,q) := f(aq) f(b) f\left(\frac{aq}{b}\right) f(abq) \quad (q)_\infty^2$$
$$= (-a^{-1}) f(a) f(b) \left(-\frac{a}{b}\right)^{-1} f\left(\frac{a}{b}\right) (-ab)^{-1} f(ab) \; (q)_\infty^2$$
$$= -a^{-3} F(a,b,q).$$