$$F\left(\tfrac{1}{a}, b, q\right) = f\left(\tfrac{1}{a}\right) f(b) f\left(\tfrac{1}{ab}\right) f\left(\tfrac{b}{a}\right) (q)_\infty^2$$

$$= (-a^{-1}) f(a) f(b) (-ab)^{-1} f(ab) \left(-\tfrac{a}{b}\right)^{-1} f\left(\tfrac{a}{b}\right) (q)_\infty^2$$

$$= (-a^{-3})\ F(a, b, q).$$

It can be shown that $F(a,b,q)$ has a Laurent series expansion

$$F(a,b,q) := \sum_{n=-\infty}^{\infty} A_n(b,q)\, a^n \qquad (\text{for } 0<|a|<\infty)$$

$$F(aq,b,q) = \sum_n A_n(b,q)\, a^n q^n =$$

$$= -a^{-3} F(a,b,q) = \sum_n -A_n\, a^{n-3}$$

$$= \sum_n -A_{n+3}\, a^n \quad \&$$

$$\sum_n A_n\, a^n q^n = -\sum_n A_{n+3}\, a^n$$

Therefore $\quad A_{n+3} = -A_n q^n$

Also,

$$F\left(\tfrac{1}{a}, b, q\right) = F(aq, b, q)$$

$$\sum_n A_{-n} a^n = \sum A_n a^{-n} = \sum A_n a^n q^n$$

$$A_{-n} = q^n A_n.$$