Conjecture:

$$\left|\{n \le N : p(n) \text{ is even}\}\right| \sim \frac{1}{2}N$$

as $N \to \infty$

Serre (1998) has proved that

$$\lim_{N\to\infty} \frac{\left|\{n \le N : p(n) \text{ is even}\}\right|}{\sqrt{N}} = +\infty$$

Conjecture: $p(n) \equiv 0 \pmod 3$ for infinitely many n.

Ono (2000) has proved that for every prime $\ell \ge 5$ there exist infinitely many pairs (A, B) such that

$$p(An + B) \equiv 0 \pmod \ell$$

for all $n \ge 0$.

Definition Let $A = \sum_{n=0}^{\infty} a_n q^n$, $B = \sum_{n=0}^{\infty} b_n q^n \in \mathbb{Z}[[q]]$

(ie power series with integer coefficients)

Let $m \ge 1$. We say

$$A \equiv B \pmod m$$

iff $a_n \equiv b_n \pmod m$ for all $n \ge 1$.

Note that this is equivalent to existence of $C = \sum_{n=0}^{\infty} c_n q^n \in \mathbb{Z}[[q]]$ such that $A = mC + B$.