$$
\begin{aligned}
q A_1(b,q^3) &= \sum_{n \equiv 2 \pmod 3} (-1)^n b^{n} q^{n(n-1)/2} \\
&= \sum_i (-1)^{3i+2} b^{3i+2} q^{(3i+2)(3i+1)/2} \\
&= q \sum_i (-1)^i b^{3i+2} q^{9i(i+1)/2},
\end{aligned}
$$

and

$$
\begin{aligned}
A_1(b,q^3) &= \sum_i (-1)^i b^{3i+2} q^{3i(i+1)/2} \\
&= \sum_{i=0}^{\infty} (-1)^i b^{3i+2} q^{3i(i+1)/2} + \sum_{i=-1}^{-\infty} (-1)^i b^{3i+2} q^{3i(i+1)/2} \\
&= \sum_{i=0}^{\infty} (-1)^i b^{3i+2} q^{3i(i+1)/2} + \sum_{i=0}^{\infty} (-1)^{i+1} b^{-3i-1} q^{3i(i+1)/2} \qquad (i \to -1-i) \\
&= \sum_{i=0}^{\infty} (-1)^i \left(b^{3i+2} - b^{-3i-1} \right) q^{3i(i+1)/2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
F(a,b,q) = {} & \sum_j (-1)^j \left(b^{-3j} - b^{3j+1} \right) q^{j(3j+1)/2} \cdot \sum_{i=0}^{\infty} (-1)^i \left(a^{-3i} - a^{+3i+3} \right) q^{3i(i+1)/2} \\
& + \sum_{i=0}^{\infty} (-1)^i \left(b^{3i+2} - b^{-3i-1} \right) q^{3i(i+1)/2} \cdot \sum_j (-1)^j \left(a^{-3j+1} - a^{3j+2} \right) q^{j(3j+1)/2} . \quad \square
\end{aligned}
$$