<u>Theorem</u> (Ramanujan)

$$p(11n+6) \equiv 0 \pmod{11} \quad \text{for } n \geq 0.$$

<u>Proof.</u>

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q)_\infty} = \frac{(q)_\infty^{10}}{(q)_\infty^{11}}$$

$$\equiv \frac{(q)_\infty^{10}}{(q^{11};q^{11})_\infty} \pmod{11}$$

Hence

$$(*) \quad \sum_{n=0}^{\infty} p(11n+6) q^{11n+6} \equiv \frac{1}{(q^{11};q^{11})_\infty} \sum_{n=0}^{\infty} p_{10}(11n+6) q^{11n+6}$$

where

$$\sum_{n=0}^{\infty} p_{10}(n) q^n = (q)_\infty^{10}$$

$$= \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} (-1)^{i+j} (2i+1)(6j+1) \left[\frac{(3i+1)(3i+2)}{2} - \frac{3j(3j+1)}{2} \right] q^{3i(i+1)/2 + j(3j+1)/2}$$

Suppose $\quad 3i(i+1)/2 + j(3j+1)/2 \equiv 6 \pmod{11}$

$$\Leftrightarrow \quad 3(i^2+i) + 3j^2 + j \equiv 12 \equiv 1 \pmod{11}$$

$$\Leftrightarrow \quad 3(i^2+12i) + 3(j^2+4j) \equiv 1 \pmod{11}$$

$$\Leftrightarrow \quad i^2+12i + j^2+4j \equiv 4 \pmod{11}$$

$$\Leftrightarrow \quad (i^2+12i+36) + (j^2+4j+4) \equiv 0 \pmod{11}$$

$$\Leftrightarrow \quad (i+6)^2 + (j+2)^2 \equiv 0 \pmod{11}$$