The Combinatorics of Ramanujan's Partition Congruences

DYSON'S RANK

Defn: Dyson (1944) defined the rank of a partition as the largest part minus the number of parts.

Example $\lambda = (7,7,6,5,3,3,1,1,1)$

$$\text{rank}(\lambda) = 7 - 9 = -2.$$

Defn: Let $N(m,n)$ denote the number of partitions of n with rank m.

Theorem: $$N(m,n) = N(-m,n)$$
for all m, n.

Proof: Let $\Omega(m,n)$ = set of partitions of n with rank m.

~~Underneath~~ Let λ be a partition.

largest part of λ' = # of parts of λ

& number of parts of λ' = largest part of λ

Hence $\text{rank}(\lambda') = -\text{rank}(\lambda)$ and the map $\lambda \mapsto \lambda'$ gives a bijection $\Omega(m,n) \to \Omega(-m,n)$.

Hence $N(m,n) = |\Omega(m,n)| = |\Omega(-m,n)| = N(-m,n)$. $\square$

NOTE: $N(m,n) = 0$ if $n < |m|$.

Defn: Let $N(m,t,n)$ = # of partitions of n with rank $\equiv m \pmod{t}$.

~~Let $t \geq 2$.~~

Theorem: $N(k,t,n) = N(t-k,t,n)$
for all k, n.

Proof: Let $t \geq 2$, $n \geq 1$.