Proof

$$\prod_{n=1}^{\infty}\frac{(1-q^n)}{(1-zq^n)(1-z^{-1}q^n)} = \prod_{n=1}^{\infty}(1-q^n)\cdot\prod_{n=1}^{\infty}\frac{1}{1-zq^n}\cdot\prod_{n=1}^{\infty}\frac{1}{1-z^{-1}q^n}$$

$$= \sum_{\pi_1\in \mathcal{PD}}(-1)^{\#(\pi_1)}q^{|\pi_1|}\sum_{\pi_2\in\mathcal{P}}z^{\#(\pi_2)}q^{|\pi_2|}\sum_{\pi_3\in\mathcal{P}}z^{-\#(\pi_3)}q^{|\pi_3|}$$

$$= \sum_{\pi=(\pi_1,\pi_2,\pi_3)\in V=\mathcal{PD}\times\mathcal{P}\times\mathcal{P}}(-1)^{\#(\pi_1)}z^{\#(\pi_2)-\#(\pi_3)}q^{|\pi_1|+|\pi_2|+|\pi_3|}$$

$$= \sum_{\pi\in V}w(\pi)\,z^{\mathrm{crank}(\pi)}q^{|\pi|}$$

$$= \sum_{n\geq 0}\sum_{m}\Bigg(\sum_{\substack{\pi\in V\\ \mathrm{crank}(\pi)=m\\ |\pi|=n}}w(\pi)\Bigg)z^m q^n$$

$$= \sum_{n\geq 0}\sum_{m}N_V(m,n)\,z^m q^n. \quad \square$$

Corollary

(1) $$N_V(m,n) = N_V(-m,n)$$
for all m, n

(2) $$N_V(k,t,n) = N_V(t-k,t,n)$$
for all k, n, t, where $t \geq 1$,

where

$$N_V(k,t,n) = \sum_{\substack{|\pi|=n\\ \mathrm{crank}(\pi)\equiv k \pmod t}} w(\pi).$$