$$\binom{p}{j} = \frac{p(p-1)\cdots(p-j+1)}{j(j-1)\cdots(2)(1)},$$

$$j!\binom{p}{j} = p(p-1)\cdots(p-j+1) \equiv 0 \pmod{p}.$$

Since p is prime & $j<p$, $p \nmid j!$.

It follows that $p \mid \binom{p}{j}$ & $\binom{p}{j} \equiv 0 \pmod{p}$.

Hence,

$$(1-q)^p \equiv (-q)^p + 1 \pmod{p}$$

$$\equiv 1 + (-1)^p q^p \pmod{p}$$

$$\equiv \begin{cases} 1 - q^p \pmod{p} & \text{if } p \text{ odd} \\ 1+q^p \equiv 1-q^p \pmod{p} & \text{if } p=2. \end{cases}$$

(2) Let $N \geq 1$.

$$(q)_\infty^p = \left((1-q)(1-q^2)\cdots(1-q^N)\right)^p + O(q^{N+1})$$

$$= (1-q)^p(1-q^2)^p\cdots(1-q^N)^p + O(q^{N+1})$$

$$\equiv (1-q^p)(1-q^{2p})\cdots(1-q^{pN}) + O(q^{N+1}) \pmod{p}$$

$$\equiv (q^p;q^p)_N + O(q^{N+1}) \pmod{p}$$

$$\equiv (q^p;q^p)_\infty + O(q^{N+1}) \pmod{p}$$

~~Result follows by letting $N \to \infty$.~~

True for all $N \geq 1$. Result follows. $\square$