Proof of (1)

We need $\displaystyle (q)_\infty = \sum_n (-1)^n q^{n(3n-1)/2}$

and the following form of JTP:

$$(zq)_\infty (z^{-1}q)_\infty (q)_\infty = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} z^{-n} \frac{(z^{2n+1}-1)}{(z-1)}$$

for $|q|<1$, $z \neq 0, 1$. (See p. 23 of Ch2 NOTES).

Let $\zeta = e^{2\pi i/5}$ so that $\zeta^5 = 1$ & $1+\zeta+\zeta^2+\zeta^3+\zeta^4 = 0$.

Recall that

$$1 - z^5 = (1-z)(1-\zeta z)(1-\zeta^2 z)(1-\zeta^3 z)(1-\zeta^4 z).$$

So that

$$(q)_\infty (\zeta q)_\infty (\zeta^2 q)_\infty (\zeta^3 q)_\infty (\zeta^4 q)_\infty$$

$$= \prod_{n=1}^{\infty} \prod_{k=0}^{4} (1-\zeta^k q^n) = \prod_{n=1}^{\infty} (1-q^{5n}) = (q^5;q^5)_\infty.$$

Next we let $z = \zeta$ in

$$\sum_{n\geq 0} \sum_m N_V(m,n) z^m q^n = \frac{(q)_\infty}{(zq)_\infty (q/z)_\infty}$$

to obtain

$$(*) \quad \sum_{n\geq 0} \sum_m N_V(m,n) \zeta^m q^n = \frac{(q)_\infty}{(\zeta q)_\infty (q/\zeta)_\infty}$$