

Now,

$$\sum_{n \geq 0} \sum_m N_v(m, n) z^m q^n$$

$$= \sum_{n \geq 0} \left(\sum_{k=0}^4 \sum_{m \equiv k \pmod{5}} N_v(m, n) z^m \right) q^n$$

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$$= \sum_{n \geq 0} \left(\sum_{k=0}^4 N_v(k, 5, n) z^k \right) q^n$$

Therefore,

$$(*) \sum_{n \geq 0} \left(\sum_{k=0}^4 N_v(k, 5, n) z^k \right) q^n = \frac{(q)_\infty}{(3q)(3q^2)} \quad \text{since } \begin{cases} z^5 = 1 \\ z^{-1} = z^4 \end{cases}$$

$$= \frac{(q)_\infty (q)_\infty (z^2)(z^3)}{(3q)(3q^2)(z^2)(z^3)(q)_\infty}$$

$$= \frac{(q)_\infty (z^2)(z^{-2})(q)_\infty}{(z^5)(z^5)} \quad \text{since } \begin{cases} z^5 = 1 \\ z^{-2} = z^3 \end{cases}$$