$$= \frac{\sum_{i} (-1)^i q^{i(3i-1)/2} \sum_{j\ge 0} (-1)^j q^{j(j+1)/2} \zeta^{-2j} \frac{(1-\zeta^{2(2j+1)})}{(1-\zeta^2)}}{(q^5;q^5)_\infty}$$

$$= \frac{\sum_{\substack{i\in\mathbb{Z}\\ j\ge 0}} (-1)^{i+j} \zeta^{-2j} \frac{(1-\zeta^{2(2j+1)})}{(1-\zeta^2)} q^{i(3i-1)/2 + j(j+1)/2}}{(q^5;q^5)_\infty}$$

We show that the coeff. of $q^{5n+4} = 0$.

$$\frac{i(3i-1)}{2} + \frac{j(j+1)}{2} \equiv 4 \pmod 5$$

$\Leftrightarrow \quad 3i^2 - i + j^2 + j \equiv 3 \pmod 5$

$\Leftrightarrow \quad 3(i^2-2i) + j^2 - 4j \equiv 3 \pmod 5$

$\Leftrightarrow \quad 3(i^2-2i+1) + (j^2-4j+4) \equiv 0 \pmod 5$

$\Leftrightarrow \quad 3(i-1)^2 + (j-2)^2 \equiv 0 \pmod 5$

$\Leftrightarrow \quad i \equiv 1 \pmod 5$ & $j \equiv 2 \pmod 5$ (EX)

In which use $2(2j+1) \equiv 2(5) \equiv 0 \pmod 5$

and $1 - \zeta^{2(2j+1)} = 0$.

It follows that the coeff of q^{5n+4} on RHS(*) $= 0$.

Hence
$$\sum_{k=0}^{4} N_V(k, 5, 5n+4)\, \zeta^k = 0,$$

and ζ is a root of the polynomial

$$p(x) = \sum_{k=0}^{4} N_V(k, 5, 5n+4)\, x^k \in \mathbb{Z}[x].$$