But the minimal polynomial of ζ (over $\mathbb{Q}$) is

$$p_5(x) = \sum_{k=0}^{4} x^k = 1+x+x^2+x^3+x^4.$$

It follows that

$$N_V(0,5,5n+4) = N_V(1,5,5n+4) = N_V(2,5,5n+4)$$
$$= N_V(3,5,5n+4) = N_V(4,5,5n+4).$$

But

$$\sum_{k=0}^{4} N_V(k,5,5n+4) = p(5n+4)$$

so that

$$5\,N_V(0,5,5n+4) = p(5n+4)$$

&

$$N_V(k,5,5n+4) = \frac{p(5n+4)}{5} \quad \text{for } 0 \le k \le 4. \ \square$$

Note: The proofs of (2), (3) are similar except that (3) requires Winquist's Identity.

Theorem (Andrews & G.)

$$M(m,n) = N_V(m,n)$$

for all $n \geq 2$ & for all m.

Proof: We need the q-binomial thm:

$$(*) \qquad \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(q)_n} = \frac{(az)_\infty}{(z)_\infty} \qquad (\text{for } |q|, |z| < 1).$$

Suppose $|q|<1$ & $|q| < |z| < 1/|q|$.