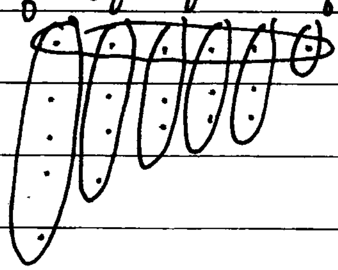
$$\frac{z^{-j} q^{j}}{(q^2;q)_{j-1}(zq^{j+1})_\infty} = \sum_{\substack{\pi \in P \\ \omega(\pi)=j}} z^{\mathrm{crank}(\pi)} q^{|\pi|}$$

and

$$\sum_{j=1}^{\infty} \frac{z^{-j} q^{j}}{(q^2;q)_{j-1}(zq^{j+1})_\infty} = \sum_{\substack{\pi \in P \\ \omega(\pi)\geq 1}} z^{\mathrm{crank}(\pi)} q^{|\pi|}$$

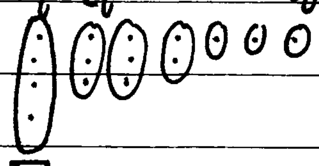
$$\frac{1}{(zq)_\infty} = \sum_{\pi \in P} z^{\ell(\pi)} q^{|\pi|}$$

$$zq^{n_1}\, zq^{n_2}\, zq^{n_3} \cdots zq^{n_k} = z^{k} q^{n_1+n_2+\cdots+n_k}$$



$$\frac{q}{(zq)_\infty} = \sum_{\pi \in P_1} z^{\ell'(\pi)} q^{|\pi|}$$

$$zq^{n_1} zq^{n_2} \cdots zq^{n_k}$$



where P_1 = set of partitions with atleast one 1 &

$\boxed{\cdot} \sim q^1$

$$\ell'(\pi) = \begin{cases} \ell(\pi) & \text{if } \pi \neq (1) \\ 0 & \text{if } \pi = (1) \end{cases}$$

Hence, $$\frac{(1-q)}{(zq)_\infty} = 1 + (z-1)q^1 + \sum_{\substack{\pi \in P \\ |\pi| \geq 2}} z^{\ell(\pi)} q^{|\pi|} - \sum_{\substack{\pi \in P_1 \\ |\pi| \geq 2}} z^{\ell(\pi)} q^{|\pi|}$$

$$= 1 + (z-1)q^1 + \sum_{\substack{\pi \in P \setminus P_1 \\ |\pi| \geq 2}} z^{\ell(\pi)} q^{|\pi|}$$