Theorem (Ramanujan)

$$p(5n+4) \equiv 0 \pmod 5$$

for all $n \geq 0$.

Proof: We need

$$(q)_\infty = \prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}$$

& $$(q)_\infty^3 = \prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n\geq 0} (-1)^n (2n+1) q^{n(n+1)/2}$$

$$\frac{1}{(q)_\infty} = \sum_{n=0}^{\infty} p(n) q^n$$

$$\frac{(q)_\infty^4}{(q)_\infty^5} = \sum_{n=0}^{\infty} p(n) q^n$$

$$(q)_\infty^5 \equiv (q^5;q^5)_\infty \pmod 5$$

so

$$\frac{1}{(q)_\infty^5} \equiv \frac{1}{(q^5;q^5)_\infty} \pmod 5$$

and $$\frac{(q)_\infty^4}{(q)_\infty^5} \equiv \frac{(q)_\infty^4}{(q^5;q^5)_\infty} \pmod 5$$

$$\frac{1}{(q^5;q^5)_\infty} = \sum_{n=0}^{\infty} p(n) q^{5n}$$

$$(q)_\infty^4 = (q)_\infty (q)_\infty^3 = \sum_{i=-\infty}^{\infty} (-1)^i q^{i(3i-1)/2} \sum_{j=0}^{\infty} (-1)^j (2j+1) q^{j(j+1)/2}$$

Hence $$\sum_{n=0}^{\infty} p(n) q^n \equiv \sum_{i=-\infty}^{\infty} (-1)^i q^{i(3i-1)/2} \sum_{j=0}^{\infty} (-1)^j (2j+1) q^{j(j+1)/2} \cdot \sum_{k=0}^{\infty} p(k) q^{5k}$$