Hence

$$p(n) \equiv \sum_{\substack{\frac{i(3i-1)}{2} + \frac{j(j+1)}{2} + 5k = n \\ j \geq 0 \text{ and } k \geq 0}} (-1)^{i+j}(2j+1)\, p(k) \pmod 5$$

$$(*) \quad p(5n+4) \equiv \sum_{\substack{\frac{i(3i-1)}{2} + \frac{j(j+1)}{2} + 5k = 5n+4 \\ j \geq 0 \;\&\; k \geq 0}} (-1)^{i+j}(2j+1)\, p(k) \pmod 5$$

Suppose $\frac{i(3i-1)}{2} + \frac{j(j+1)}{2} + 5k = 5n+4$ where $j \geq 0$ & $k \geq 0$.
Then

$$\frac{i(3i-1)}{2} + \frac{j(j+1)}{2} \equiv 4 \pmod 5$$

$$3i^2 - i + j^2 + j \equiv 8 \equiv 3 \pmod 5$$

$$3(i^2 - 2i) + j^2 - 4j \equiv 3 \pmod 5$$

$$3(i^2 - 2i + 1 - 1) + (j^2 - 4j + 4 - 4) \equiv 3 \pmod 5$$

$$3((i-1)^2 - 1) + (j-2)^2 - 4 \equiv 3 \pmod 5$$

$$3(i-1)^2 + (j-2)^2 \equiv 0 \pmod 5$$

$$(i-1)^2 \equiv 0^2, (\pm 1)^2, (\pm 2)^2 \equiv 0, 1, 4 \pmod 5$$

$$\cancel{(j-2)^2 \equiv 0, 1, 4 \pmod 5}$$

$$3(i-1)^2 \equiv 0, 3, 2 \pmod 5$$

$$(j-2)^2 \equiv 0, 1, 4 \pmod 5$$

$$3(i-1)^2 + (j-2)^2 \equiv 0 \pmod 5$$

iff $3(i-1)^2 \equiv 0$ & $(j-2)^2 \equiv 0 \pmod 5$

ie $i \equiv 1$ & $j \equiv 2 \pmod 5$

Hence $(2j+1) \equiv 4 + 1 \equiv 0 \pmod 5$ & by $(*)$

$$p(5n+4) \equiv 0 \pmod 5. \qquad \blacksquare$$