Roots of Unity

Let $\theta \in \mathbb{R}$.

$$e^{i\theta} := \cos\theta + i\sin\theta.$$

Then $\quad e^{i(\theta_1+\theta_2)} = e^{i\theta_1} e^{i\theta_2}$

and $\quad \left(e^{in\theta}\right) = \left(e^{i\theta}\right)^n \qquad$ (de Moivre)

$$\cos n\theta + i\sin n\theta = (\cos(\theta) + i\sin\theta)^n,$$

for $n \in \mathbb{Z}$.

Theorem Let n be a positive integer. The equation

$$z^n = 1$$

has n complex roots

$$z_k = e^{2\pi i \frac{k}{n}} \qquad k = 0, 1, 2, \ldots, n-1$$

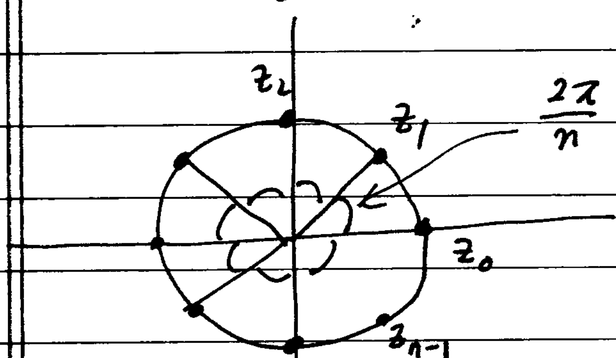
i.e.

$$z_k = \zeta^k, \quad k = 0, 1, 2, \ldots, n-1$$

where $\zeta = e^{2\pi i/n} = \cos(\pi/n) + i\sin(\pi/n)$.

Further, $\quad 1 + \zeta + \cdots + \zeta^{n-1} = 1$

and $\quad (z^n - 1) = (z-1)(z-\zeta)(z-\zeta^2)\cdots(z-\zeta^{n-1})$.



Proof

Let $z_k = e^{2\pi i k/n} \quad 0 \le k \le n-1$.

$$z_k^n = e^{2\pi i k \frac{n}{n}} = e^{2\pi i k}$$

$$= \cos(2\pi k) + i\sin(2\pi k)$$

$$= \cos(0) + i\sin(0) = 1.$$

It can be shown that the z_k, $\quad 0 \le k \le n-1$ are distinct and account for all the roots. It follows that

$$(z^n - 1) = (z-1)(z-\zeta)\cdots(z-\zeta^{n-1}).$$

Also $\quad (z^n - 1) = (z-1)(1 + z + \cdots + z^{n-1})$.

$$0 = (\zeta^n - 1) = (\zeta - 1)(1 + \zeta + \cdots + \zeta^{n-1}).$$