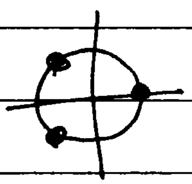
Since $\zeta \neq 1$, we have

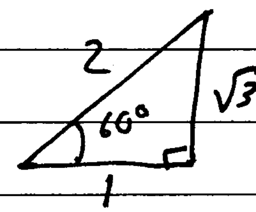
$$1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1} = 0. \qquad \square$$

<u>Note</u> ζ is called an n^{th} root of unity.

<u>Example</u>: Solve $z^3 = 1$.

$$z_k = \cos\left(\frac{2\pi k}{3}\right) + i\sin\left(\frac{2\pi k}{3}\right), \quad k = 0, 1, 2.$$





$$\begin{aligned}
z_0 &= 1 \\
z_1 &= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) \\
&= \cos(120^\circ) + i\sin(120^\circ) \\
&= \cos(180^\circ - 60^\circ) + i\sin(180^\circ - 60^\circ) \\
&= -\cos 60^\circ + i\sin 60^\circ \\
&= -\tfrac{1}{2} + i\sqrt{3}/2 \\
z_2 &= \cos(240^\circ) + i\sin(240^\circ) \\
&= -\tfrac{1}{2} - i\sqrt{3}/2
\end{aligned}$$

<u>Theorem</u> (Ramanujan)

$$\sum_{n=0}^{\infty} p(5n+4)\, q^n = 5 \prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)^6}$$

<u>Proof</u>:

$$(q)_\infty = \prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}$$

n	$n(3n-1)/2 \pmod 5$
0	0
1	$1(2)/2 \equiv 1$
2	$2(5)/2 \equiv 0$
3	$3(8)/2 = 12 \equiv 2$
4	$(-1)(-4)/2 \equiv 2$