Chapter 4 Restricted Partitions & Permutations

[See Ch3 of TEXT]

Let $p(N,M,n)$ = # of ptns of n into at most M parts each part $\leq N$.

If π is such a ptn then $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$, $m \leq M$, and $|\pi| = n = \sum_{i=1}^{m} \pi_i \leq MN$.

It follows that

$$p(N,M,n) = 0 \quad \text{if } n > MN$$

and $p(N,M,MN) = 1$ (there is only one ptn of MN into at most M parts, each part $\leq N$ namely $\pi = (\underbrace{N, N, \ldots, N}_{M\text{-times}})$).

Therefore,

$$G(N,M;q) = \sum_{n \geq 0} p(N,M,n) q^n = \sum_{\pi \in \mathcal{B}} q^{|\pi|} \quad \text{is a polynomial}$$

in q of degree NM.

<u>Theorem</u> For $M, N \geq 0$,

$$(*) \quad G(N,M;q) = \frac{(1-q^{N+M})(1-q^{N+M-1})\cdots(1-q^{M+1})}{(1-q^{N})(1-q^{N-1})\cdots(1-q^{1})}$$

$$= \frac{(q)_{M+N}}{(q)_M (q)_N}.$$

<u>Proof:</u> Let $g(N,M;q)$ denote the RHS of $(*)$.

Firstly,

$$\frac{(1-q^{M+N})}{(1-q^{N})} \cdots \frac{(1-q^{M+1})}{(1-q^{1})} = \frac{(1-q^{M+N})\cdots(1-q^{M+1})(1-q^{M})\cdots(1-q)}{(q)_N (q)_M}$$

$$= \frac{(q)_{M+N}}{(q)_M (q)_N}.$$

(2)

$$(**)\quad g(N,0;q) = \frac{(q)_N}{(q)_0\,(q)_N} = \frac{(q)_N}{(q)_N} = 1, \quad \&$$

similarly $g(0,M;q) = 1.$

Suppose N & $M \geq 1.$

$$g(N,M;q) - g(N,M-1;q) = \frac{(q)_{M+N}}{(q)_M\,(q)_N} - \frac{(q)_{M+N-1}}{(q)_{M-1}\,(q)_N}$$

$$= \frac{(q)_{M+N-1}}{(q)_M\,(q)_N}\Big((1-q^{M+N}) - (1-q^M) \Big)$$

$$= \frac{(q)_{M+N-1}}{(q)_M\,(q)_N}\left(q^M - q^{M+N} \right) = q^M \frac{(1-q^N)(q)_{M+N-1}}{(q)_M\,(q)_N}$$

$$= q^M \frac{(q)_{M+N-1}}{(q)_M\,(q)_{N-1}} = q^M g(N-1,M;q) \quad \&$$

$$(***)\quad g(N,M;q) = g(N,M-1;q) + q^M g(N-1,M;q).$$

Equations $(**)$, $(***)$ uniquely define $g(N,M;q)$ for all $M, N \geq 0$.

The empty partition $()$ of 0 is the only ptn with no parts & the only ptn whose largest part is 0. Hence,

$$p(N,0,n) = p(0,M,n) = \begin{cases} 1 & \text{if } N=M=n=0 \\ 0 & \text{otherwise} \end{cases}$$

Hence, (3)

$$G(N,0;q) = \sum_{n\geq 0} p(N,0,n)\, q^n = 1$$

&

$$G(0,M;q) = \sum_{n\geq 0} p(0,M,n)\, q^n = 1,$$

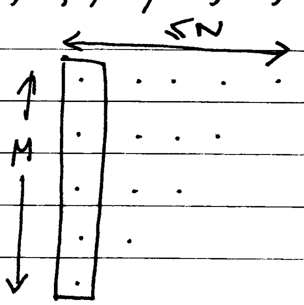
for all $M, N \geq 0$.

Suppose M & $N \geq 1$.

$p(N,M,n)$ = # of ptns of n into at most M parts $\leq N$

$p(N,M-1,n)$ = ... $M-1$ . . . .

$p(N,M,n) - p(N,M-1,n)$ = # of ptns of n into exactly M parts each $\leq N$



If we delete one from each part we obtain a ptn of $n-M$ into at most M parts each part $\leq N-1$. Conversely this is reversible by adding one to each part.

Therefore

$$p(N,M,n) - p(N,M-1,n) = \begin{cases} p(N-1, M, n-M) & \text{if } n \geq M \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{n\geq 0} p(N,M,n)\, q^n - \sum_{n\geq 0} p(N,M-1,n)\, q^n = \sum_{n\geq M} p(N-1,M,n-M)\, q^n$$

$$G(N,M;q) - G(N,M-1;q) = q^M \sum_{n\geq M} p(N-1,M,n-M)\, q^{n-M}$$

$$= q^M \sum_{n\geq 0} p(N-1,M,n)\, q^n$$

$$= q^M\, G(N-1,M;q).$$

$$\frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-m+1})}{(1-q^m)(1-q^{m-1})\cdots(1-q^1)}\;\frac{\cancel{(1-q^{n-m})\cdots(1-q)}}{\cancel{(1-q^{n-m})\cdots(1-q)}}$$

Thus $g(N,M;q)$ & $G(N,M;q)$ *satisfy the* (4)
same initial conditions (**) & *the same recurrence* (***).
By uniqueness,

$$G(N,M;q) = g(N,M;q) = \frac{(q)_{M+N}}{(q)_M (q)_N}. \quad \square$$

GAUSSIAN POLYNOMIALS

The polynomials

$$G(N,M;q) = \sum_{n\ge 0} p(N,M,n)q^n = \frac{(q)_{M+N}}{(q)_M(q)_N}$$

are called ***Gaussian polynomials*** *or* ***q-binomial coefficients***

Definition: *The Gaussian polynomial*

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ m \end{bmatrix} := \begin{cases} \dfrac{(q)_n}{(q)_m (q)_{n-m}} & \text{if } 0\le m\le n \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$\begin{bmatrix} N \\ M \end{bmatrix} = G(N-M,M;q) \quad \text{if } 0\le M\le N.$$

Theorem *Let* $0\le m\le n$ *be integers. The Gaussian poly* $\begin{bmatrix} n \\ m \end{bmatrix}$ *is a poly. in* q *of degree* $m(n-m)$ *satisfying*

(1) $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1,$

(2) $\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ n-m \end{bmatrix}$

(5)

(3) $$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \quad (\text{for } n, m \geq 1),$$

(4) $$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^{m} \begin{bmatrix} n-1 \\ m \end{bmatrix} \quad (\text{for } n, m \geq 1),$$

(5) $$\lim_{q \to 1} \begin{bmatrix} n \\ m \end{bmatrix} = \frac{n!}{m!(n-m)!} = \binom{n}{m}.$$

Proof: $\begin{bmatrix} n \\ m \end{bmatrix} = \sum_{j \geq 0} p(n-m, m, j) q^j = G(n-m, m; q)$ which is a polyn of degree $m(n-m)$.

Suppose $1 \leq m < n$ are integers.

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \frac{(q)_n}{(q)_0 (q)_n} = \frac{(q)_n}{(q)_n} = 1$$

$$\begin{bmatrix} n \\ n \end{bmatrix} = \frac{(q)_n}{(q)_n (q)_0} = \frac{(q)_n}{(q)_n} = 1.$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(q)_n}{(q)_m (q)_{n-m}} = \frac{(q)_n}{(q)_{n-m} (q)_{n-(n-m)}} = \begin{bmatrix} n \\ n-m \end{bmatrix}$$

Suppose $m, n \geq 1$.

$$\begin{bmatrix} n \\ m \end{bmatrix} - \begin{bmatrix} n-1 \\ m \end{bmatrix} = \frac{(q)_n}{(q)_m (q)_{n-m}} - \frac{(q)_{n-1}}{(q)_m (q)_{n-m-1}}$$

$$= \frac{(q)_{n-1}}{(q)_m (q)_{n-m}} \left((1-q^n) - (1-q^{n-m}) \right)$$

$$= \frac{(q)_{n-1}}{(q)_m (q)_{n-m}} \left(q^{n-m} - q^n \right) = q^{n-m} \frac{(1-q^m)(q)_{n-1}}{(q)_m (q)_{n-m}}$$

$$= q^{n-m} \frac{(q)_{n-1}}{(q)_{m-1} (q)_{n-m}} = q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}$$ which is (3).

By (3) $\begin{bmatrix} n \\ n-m \end{bmatrix} = \begin{bmatrix} n-1 \\ n-m \end{bmatrix} + q^{n-(n-m)} \begin{bmatrix} n-1 \\ (n-m)-1 \end{bmatrix}$

& $\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^{m} \begin{bmatrix} n-1 \\ m \end{bmatrix}$ by (2).

Recall, $\lim\limits_{q\to 1} \dfrac{1-q^j}{1-q} = \lim\limits_{q\to 1} 1+q+\cdots+q^{j-1} = j.$

$$\lim_{q\to 1} \begin{bmatrix} n \\ m \end{bmatrix} = \lim_{q\to 1} \frac{(q)_n}{(q)_m (q)_{n-m}} = \lim_{q\to 1} \frac{(q)_n \frac{1}{(1-q)^n}}{(q)_m (q)_{n-m} \frac{1}{(1-q)^m (1-q)^{n-m}}}$$

$$= \lim_{q\to 1} \frac{\left(\frac{(1-q)}{(1-q)}\frac{(1-q^2)}{(1-q)}\cdots\frac{(1-q^n)}{(1-q)}\right)}{\left(\frac{(1-q)}{(1-q)}\frac{(1-q^2)}{(1-q)}\cdots\frac{(1-q^m)}{(1-q)}\right)\left(\frac{(1-q)(1-q^2)\cdots(1-q^{n-m})}{(1-q)(1-q)\cdots(1-q)}\right)}$$

$$= \frac{(1)(2)\cdots(n)}{(1)(2)\cdots(m)(1)(2)\cdots(n-m)} = \frac{n!}{m!\,(n-m)!} = \binom{n}{m}. \;]$$

<u>Theorem</u>

(1) $(z)_N = \sum\limits_{j=0}^{N} (-1)^j z^j q^{j(j-1)/2} \begin{bmatrix} N \\ j \end{bmatrix}$ for $N \geq 0$;

(2) $\dfrac{1}{(z)_N} = \sum\limits_{j=0}^{\infty} \begin{bmatrix} N+j-1 \\ j \end{bmatrix} z^j$ for $N \geq 1$, ($|q|<1$ & $|z|<1$)

<u>Proof:</u>

(1) Clearly true for $N=0$. Suppose $N \geq 1$.

$$(z)_N = (1-z)(1-zq)\cdots(1-zq^{N-1}) = \frac{(z)_\infty}{(zq^N)_\infty}$$

(7)

We need q-bin. Thm:

$$\sum_{n=0}^{\infty} \frac{(a)_n\, t^n}{(q)_n} = \frac{(at)_\infty}{(t)_\infty} \qquad \text{for } |q|,|t|<1.$$

Let $t = zq^N$, $at = z$ ie $a = q^{-N}$.

So

$$(z)_N = \sum_{n=0}^{\infty} \frac{(q^{-N})_n}{(q)_n} (zq^N)^n \qquad \text{(assuming } |q|<1 \text{ \& } |z|<1\text{).}$$

$$(q^{-N})_n = (1-q^{-N})(1-q^{-N+1})\cdots(1-q^{-N+n-1}) = 0 \text{ if } n>N.$$

if $n \le N$

$$(q^{-N})_n = (-q^{-N})(1-q^N)(-q^{-N+1})(1-q^{N-1}) \cdots (-q^{-N+n-1})(1-q^{N+n-1})$$

$$= (-1)^n q^{\frac{n(n-1)}{2} - Nn} (1-q^N)(1-q^{N-1})\cdots(1-q^{N+n-1})$$

&

$$z^n q^{Nn} \frac{(q^{-N})_n}{(q)_n} = (-1)^n q^{n(n-1)/2} z^n \frac{(1-q^N)(1-q^{N-1})\cdots(1-q^{N+n-1})}{(1-q^n)(1-q^{n-1})\cdots(1-q)}$$

$$= \begin{bmatrix} N \\ n \end{bmatrix} (-1)^n z^n q^{n(n-1)/2} \qquad \text{if } n \le N.$$

Hence,

$$(z)_n = \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} (-1)^n z^n q^{n(n-1)/2}$$

for $|z|, |q| < 1$ & for all z, q by analy. continuation.

(2). Now suppose $|q|, |z| < 1$ & $N \geq 1$.

$$\frac{1}{(z)_N} = \frac{1}{(1-z)(1-zq)\cdots(1-zq^{N-1})} = \frac{(zq^N)_\infty}{(z)_\infty}$$

$$= \sum_{j=0}^{\infty} \frac{(q^N)_j}{(q)_j} z^j$$ (by q-bin with $a = q^N$ & $t = z$)

$$= \sum_{j=0}^{\infty} \frac{(1-q^N)(1-q^{N+1})\cdots(1-q^{N+j})}{(1-q)(1-q^2)\cdots(1-q^j)} z^j$$

$$= \sum_{j=0}^{\infty} \begin{bmatrix} N+j-1 \\ j \end{bmatrix} z^j .$$

Theorem

(1) $$\sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix} = \begin{cases} (q;q^2)_n & \text{if } m = 2n \\ 0 & \text{otherwise} \end{cases}$$

(2) $$\sum_{j=0}^{n} q^j \begin{bmatrix} m+j \\ m \end{bmatrix} = \begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix} \quad \text{if } m,n \geq 0$$

(3) $$\sum_{k=0}^{h} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ h-k \end{bmatrix} q^{(n-k)(h-k)} = \begin{bmatrix} m+n \\ h \end{bmatrix}$$

(4) $$\sum_{r \geq 0} \begin{bmatrix} M-m \\ r \end{bmatrix} \begin{bmatrix} N+m \\ m+r \end{bmatrix} \begin{bmatrix} m+n+r \\ M+N \end{bmatrix} q^{(N-r)(M-r-m)} = \begin{bmatrix} m+n \\ M \end{bmatrix} \begin{bmatrix} n \\ N \end{bmatrix}$$

Proof (9)

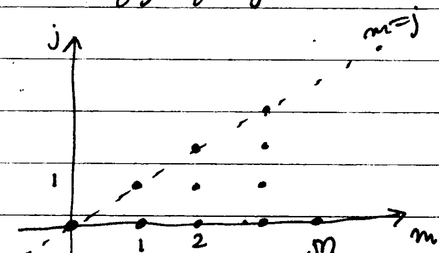
(1) Let $f(m) := \sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix}$ for $m \geq 0$.

Then $\sum_{m=0}^{\infty} \frac{f(m)\, z^m}{(q)_m} = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \frac{(-1)^j\, z^m}{(q)_j\, (q)_{m-j}}$ (assume $0<|z|<|q|<1$)

$$= \sum_{j=0}^{\infty} \sum_{m=j}^{\infty} \frac{(-1)^j\, z^m}{(q)_j\, (q)_{m-j}} \quad \text{(by abs. conv.)}$$

$(m=j+k)$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^j\, z^{j+k}}{(q)_j\, (q)_k}$$



$\{(m,j) : m \geq 0 \;\&\; 0 \leq j \leq m\}$
$= \{(m,j) : j \geq 0 \;\&\; m \geq j\}$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{(q)_j} \sum_{k=0}^{\infty} \frac{z^k}{(q)_k}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{(q)_j} \, \frac{1}{(z)_\infty} \quad \text{(by Euler's cor. to } q\text{-bin thm.)}$$

$$= \frac{1}{(-z)_\infty} \frac{1}{(z)_\infty} = \frac{1}{(z^2;q^2)_\infty} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(q^2;q^2)_k}$$

$$\sum_{m=0}^{\infty} \frac{f(m)\, z^m}{(q)_m} = \sum_{m=0}^{\infty} \frac{f(2m)\, z^{2m}}{(q)_{2m}} + \sum_{m=0}^{\infty} \frac{f(2m+1)\, z^{2m+1}}{(q)_{2m+1}}.$$

Hence

$$\frac{f(2m)}{(q)_{2m}} = \frac{1}{(q^2;q^2)_m}, \quad \& \; f(2m) = \frac{(1-q)(1-q^2)\cdots(1-q^{2m})}{(1-q^2)(1-q^4)\cdots(1-q^{2m})}$$

$$= (1-q)(1-q^3)\cdots(1-q^{2m-1}) = (q;q^2)_m$$

(10)

and $f(m) = 0$ if m is odd. □

(2) We prove (2) by induction on n. Result is clearly true for $n=0$ since $\begin{bmatrix} m \\ m \end{bmatrix} = 1 = \begin{bmatrix} m+1 \\ m+1 \end{bmatrix}$.
Assume result is true for a given n.

$$\begin{bmatrix} n+m+2 \\ m+1 \end{bmatrix} = \begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix} + q^{n+1} \begin{bmatrix} n+m+1 \\ m \end{bmatrix}$$

$$= \sum_{j=0}^{n} q^{j} \begin{bmatrix} m+j \\ m \end{bmatrix} + q^{n+1} \begin{bmatrix} n+m+1 \\ m \end{bmatrix}$$

$$= \sum_{j=0}^{n+1} q^{j} \begin{bmatrix} m+j \\ m \end{bmatrix}$$ & result is true for $n+1$,

and true in general for $n \geq 0$ by induction.

(3) [q-analog of Chu-Vandermonde summation]
Let $m, n \geq 0$.

$$(z)_{m+n} = (1-z)(1-zq)\cdots(1-zq^{n-1})(1-zq^{n})\cdots(1-zq^{n+m-1})$$

(*) $$(z)_{m+n} = (z)_n\,(zq^n;q)_m$$

$$(z)_{m+n} = \sum_{j=0}^{m+n} \begin{bmatrix} m+n \\ j \end{bmatrix} (-1)^j z^j q^{j(j-1)/2}$$

$$(z)_n\,(zq^n)_m = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k z^k q^{k(k-1)/2}$$

$$\cdot \sum_{l=0}^{m} \begin{bmatrix} m \\ l \end{bmatrix} (-1)^l (zq^n)^l q^{l(l-1)/2}$$

$$= \sum_{h=0}^{m+n} (-1)^h z^h q^{h(h-1)/2} \sum_{k=0}^{h} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ h-k \end{bmatrix} q^{(n-k)(h-k)}.$$

(11)

since $k + l = h$

$$nl + \frac{k(k-1)}{2} + \frac{l(l-1)}{2} = \frac{k(k-1)}{2} + \frac{(h-k)(h-k-1)}{2} + n(h-k)$$

$$= \frac{k(k-1)}{2} - \frac{h(k+1)}{2} - \frac{kh}{2} + \frac{h^2}{2} + \frac{k(k+1)}{2} + n(h-k)$$

$$= k^2 - hk + \frac{h^2}{2} + nh - nk - h/2$$

$$= \frac{h(h-1)}{2} + k^2 - hk + nh - nk$$

$$= \frac{h(h-1)}{2} + (n-k)(h-k).$$

The result follows by comparing the coeff of z^h on both sides of $(*)$.

(4) By Heine's transformation

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n t^n}{(q)_n (c)_n} = \frac{(b)_\infty (at)_\infty}{(c)_\infty (t)_\infty} \sum_{n=0}^{\infty} \frac{(c/b)_n (t)_n b^n}{(at)_n (q)_n}$$

$$= \frac{(b)_\infty (at)_\infty}{(c)_\infty (t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n (c/b)_n b^n}{(at)_n (q)_n}$$

$$= \frac{\cancel{(b)_\infty} \cancel{(at)_\infty}}{(c)_\infty (t)_\infty} \cdot \frac{(c/b)_\infty (bt)_\infty}{\cancel{(at)_\infty} \cancel{(b)_\infty}} \sum_{n=0}^{\infty} \frac{(abt/c)_n (b)_n (c/b)^n}{(bt)_n (q)_n}$$

(by Heine's trf)

$$= \frac{(c/b)_\infty (bt)_\infty}{(c)_\infty (t)_\infty} \sum_{n=0}^{\infty} \frac{(b)_n (abt/c)_n (c/b)^n}{(bt)_n (q)_n}$$

$$= \frac{\cancel{(c/b)_\infty} \cancel{(bt)_\infty}}{\cancel{(c)_\infty} (t)_\infty} \cdot \frac{(abt/c)_\infty \cancel{(c)_\infty}}{\cancel{(bt)_\infty} \cancel{(c/b)_\infty}} \sum_{n=0}^{\infty} \frac{(c/a)_n (c/b)_n (abt/c)^n}{(C)_n (q)_n}$$

(by Heine)

Hence

$$\frac{(t)_\infty}{(abt/c)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n t^n}{(q)_n (c)_n} = \sum_{n=0}^{\infty} \frac{(c/a)_n (c/b)_n \left(\frac{abt}{c}\right)^n}{(c)_n (q)_n} \qquad (12)$$

$$\cdot \sum_{j=0}^{\infty} \frac{\left(\frac{c}{ab}\right)_j \left(\frac{abt}{c}\right)^j}{(q)_j} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n t^n}{(q)_n (c)_n} = \sum_{N=0}^{\infty} \frac{\left(\frac{c}{a}\right)_N \left(\frac{c}{b}\right)_N \left(\frac{abt}{c}\right)^N}{(c)_N (q)_N}$$

Comparing coeff of t^N on both sides:

$j+n=N$

$$\sum_{n=0}^{N} \frac{(a)_n (b)_n \left(\frac{c}{ab}\right)_{N-n} \left(\frac{ab}{c}\right)^{N-n}}{(q)_n (c)_n (q)_{N-n}} = \frac{\left(\frac{c}{a}\right)_N \left(\frac{c}{b}\right)_N \left(\frac{ab}{c}\right)^N}{(c)_N (q)_N}$$

We multiply both side by $\frac{(q)_N}{\left(\frac{c}{ab}\right)_N} \frac{c^N}{a^N b^N}$

to obtain

$$\sum_{n=0}^{N} \frac{(a)_n (b)_n}{(q)_n (c)_n} \frac{(q)_N}{(q)_{N-n}} \frac{\left(\frac{c}{ab}\right)_{N-n}}{\left(\frac{c}{ab}\right)_N} \left(\frac{c}{ab}\right)^n = \frac{\left(\frac{c}{a}\right)_N \left(\frac{c}{b}\right)_N}{(c)_N (q)_N}$$

$$(a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

$$= (-a)(1-a^{-1})(-aq)(1-a^{-1}q^{-1})\cdots(-aq^{n-1})(1-a^{-1}q^{1-n})$$

$$= (-a)^n q^{n(n-1)/2} \left(\frac{q^{1-n}}{a}\right)_n$$

$$\frac{(x)_{N-n}}{(x)_N} = \frac{1}{(1-xq^{N-n})\cdots(1-xq^{N-1})} = \frac{1}{(xq^{N-n})_n}$$

(13)

$$= \frac{1}{\left(\frac{q^{1-n}}{x q^{N-n}}\right)_m \left(-x q^{N-n}\right)^n q^{\binom{n}{2}}} = \frac{1}{\left(q^{1-N} x^{-1}\right)_n \left(-x q^{N-n}\right)^n q^{\binom{n}{2}}}$$

Therefore,

$$\frac{(q)_N}{(q)_{N-n}} \cdot \frac{\left(\frac{c}{ab}\right)_{N-n}}{\left(\frac{c}{ab}\right)_N} \left(\frac{c}{ab}\right)^n$$

$$= \frac{\left(q^{-N}\right)_m (-q)^n \left(\frac{c}{ab}\right)^n}{\left(q^{1-N} a b c^{-1}\right)_m \left(-\frac{c}{ab}\right)^n}$$

$$= \frac{\left(q^{-N}\right)_n q^n}{\left(q^{1-N} a b c^{-1}\right)_m}.$$

Hence

$$\sum_{n=0}^{N} \frac{(a)_n (b)_n \left(q^{-N}\right)_n q^n}{(c)_n \left(a b c^{-1} q^{1-N}\right)_n (q)_n} = \frac{(c/a)_N (c/b)_N}{(c)_N (c/ab)_N}$$

(Jackson's q-analog of Saalschütz's Thm).

Now let

$$a = q^{-M+m}, \quad b = q^{m+n+1}, \quad c = q^{m+1}$$

$$\sum_{r=0}^{N} \frac{\left(q^{-M+m}\right)_r \left(q^{m+n+1}\right)_r q^r \left(q^{-N}\right)_r}{\left(q^{m+1}\right)_r \left(q^{-N-M+m+n+1}\right)_r (q)_r} = \frac{\left(q^{M+1}\right)_N \left(q^{-n}\right)_N}{\left(q^{m+1}\right)_N \left(q^{M-n}\right)_N}$$

We use

$$\left(q^{-N}\right)_r = \frac{(q)_N}{(q)_{N-r}} (-1)^r q^{r(r-1)/2 - rN}$$

(14)

and $\quad (q^{m+1})_r = \dfrac{(q)_{m+r}}{(q)_m}$.

Thus

$$\sum_{r=0}^{N} \frac{(q)_{M-m}}{(q)_{M-m-r}} \frac{(q)_{m+n+r}}{(q)_{m+n}} \frac{(q)_N}{(q)_{N-r}} \frac{(q)_m}{(q)_{m+r}} \frac{(q)_{m+n-M-N}}{(q)_{m+n-M-N+r}} \frac{1}{(q)_r} \cdot q^{(r^2+rm-rM-rN)}$$

$$= \frac{(q)_{M+N}}{(q)_M} \frac{(q)_n}{(q)_{n-N}} \frac{(q)_m}{(q)_{m+N}} \frac{(q)_{n+m-M-N}}{(q)_{n+m-M}} q^{(Nm-NM)}$$

and

$$\sum_{r\geq 0} \left(\frac{(q)_{M-m}}{(q)_{M-m-r}(q)_r}\right)\left(\frac{(q)_{N+m}}{(q)_{m+r}(q)_{N-r}}\right)\left(\frac{(q)_{m+n+r}}{(q)_{M+N}(q)_{m+n+r-M-N}}\right) \cdot q^{(N-r)(M-m-r)}$$

$$= \left(\frac{(q)_{m+n}}{(q)_M (q)_{m+n-M}}\right)\left(\frac{(q)_n}{(q)_N (q)_{n-N}}\right). \quad \square$$

(15)

Multinomial coefficient:

Note $\binom{n}{r} = \binom{n}{r, n-r}$

Let $n_1, n_2, \ldots, n_k \geq 0$.

$$\binom{n_1+n_2+\cdots+n_k}{n_1, n_2, \ldots, n_k} := \frac{(n_1+n_2+\cdots+n_k)!}{n_1!\, n_2! \cdots n_k!}$$

Multinomial Thm:

$$(a_1+a_2+\cdots+a_k)^n = \sum_{\substack{n_1, n_2, \ldots, n_k \geq 0 \\ n_1+n_2+\cdots+n_k = n}} \binom{n}{n_1, n_2, \ldots, n_k} a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}.$$

q-Multinomial Coefficient (or Gaussian multinomial coeff.)

$$\begin{bmatrix} n_1+n_2+\cdots+n_k \\ n_1, n_2, \ldots, n_k \end{bmatrix} := \frac{(q)_{n_1+n_2+\cdots+n_k}}{(q)_{n_1} (q)_{n_2} \cdots (q)_{n_k}}.$$

Definition A multiset is a "set" of possibly repeated elements.

Eg $\{1,1,1,2,2,3,3,3,3\} = \{1^3, 2^2, 3^4\}$.

A permutation of a multiset is a just a rearrangement of the elements. For example

$1\,2 \;\; 1\,3\;\;3 \;\; 2\,3\,1\,3$ is a permutation of $1\,1\,1\,2\,2\,3\,3\,3\,3$

Defn Let $m = m_1+m_2+\cdots+m_r$.

Let $inv(m_1, m_2, \ldots, m_r; n)$ = # of permutations $x_1, x_2, \ldots, x_m$ of the multiset $\{1^{m_1}, 2^{m_2}, \ldots, r^{m_r}\}$ in which there are exactly n pairs (x_i, x_j) where $i<j$ but $x_i > x_j$.

(such a pair is called an inversion)

(16)

Example $m_1 = 1,\ m_2 = 2,\ m_3 = 1$

Permutation of 1, 2, 2, 3	# of inversions
1, 2, 2, 3	0
1, 2, 3, 2	1
1, 3, 2, 2	2
2, 1, 2, 3	1
2, 1, 3, 2	2
2, 2, 1, 3	2
2, 2, 3, 1	3
2, 3, 1, 2	3
2, 3, 2, 1	4
3, 1, 2, 2	3
3, 2, 1, 2	4
3, 2, 2, 1	5

$inv(1,2,2,3;\ 0) = 1$

$inv(1,2,2,3;\ 1) = 2$

$inv(1,2,2,3;\ 2) = 3$

$inv(1,2,2,3;\ 3) = 3$

$inv(1,2,2,3,\ 4) = 2$

$inv(1,2,2,3,\ 5) = 1$

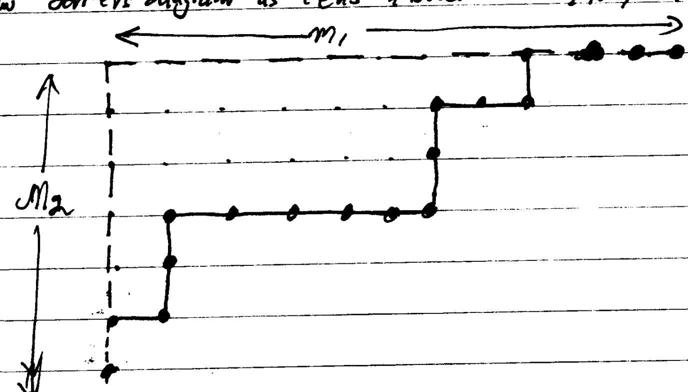
<u>Theorem</u> (17)

$$\mathrm{inv}(m_1, m_2; n) = p(m_1, m_2, n)$$

(the # of partitions of n into at most m_2 parts each part $\le m_1$).

<u>Proof:</u> We describe a bijection between the permutations of $1^{m_1}2^{m_2}$ with n inversions & the partitions of n into at most m_2 parts each part $\le m_1$

Suppose λ is a partition with each part $\le m_1$ & # of parts $\le m_2$.
We draw Ferrers diagram as cells inside a $m_2 \times m_1$ box:



Eg. $(8, 6, 6, 1, 1)$

We follow a path starting at $(0, m_1)$ & finishing at $(m_2, 0)$ following the boundary of the partition. We form a sequence of 1s & 2s — 2 if we move vertically & 1 if we move horizontally.

Eg: 1 1 1 2 1 1 2 2 1 1 1 1 1 2 2 1 2 2

This gives a permutation of $\{1^{m_1}, 2^{m_2}\}$.

largest part = # of 1s to right of the first 2
second largest part = ... second 2
etc
Thus the # of inversions = sum of parts of the partition.
This gives the desired bijection. The process can be reversed by drawing the boundary from the seq. of 1s & 2s. □

Corollary

$$\sum_{n\geq 0} \mathrm{inv}(m_1, m_2; n)\, q^n = \begin{bmatrix} m_1+m_2 \\ m_1 \end{bmatrix} = \frac{(q)_{m_1+m_2}}{(q)_{m_1}(q)_{m_2}}.$$

Theorem (P.A. MacMahon)

$$\sum_{n\geq 0} \mathrm{inv}(m_1, m_2, \dots, m_k; n)\, q^n = \begin{bmatrix} m_1+m_2+\dots+m_k \\ m_1, m_2, \dots, m_k \end{bmatrix}.$$

The Major Index

For a permutation $x_1, x_2, \dots, x_m$ of the multiset $\{1^{m_1} 2^{m_2} \cdots r^{m_r}\}$ define

$$\chi(x_i) = \begin{cases} i & \text{if } x_i > x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

The greater index of $x_1 x_2 \dots x_m = \sum_{i=1}^{m-1} \chi(x_i)$
where $m = m_1 + \dots + m_r$.

<u>Example</u> $m_1 = 1,\ m_2 = 2,\ m_3 = 1$

Perms of 1, 2, 2, 3	Greater index
1, 2, 2, 3	0
1, 2, 3, 2	3
1, 3, 2, 2	2
2, 1, 2, 3	1
2, 1, 3, 2	4
2, 2, 1, 3	2
2, 2, 3, 1	3
2, 3, 1, 2	2
2, 3, 2, 1	5
3, 1, 2, 2	1
3, 2, 1, 2	3
3, 2, 2, 1	4

$\mathrm{ind}(1,2,1;0) = 1$

$\mathrm{ind}(1,2,1;1) = 2$

$\mathrm{ind}(1,2,1;2) = 3$

$\mathrm{ind}(1,2,1;3) = 3$

$\mathrm{ind}(1,2,1;4) = 2$

$\mathrm{ind}(1,2,1;5) = 1.$

<u>Theorem</u> $$\sum_{n\geq 0} \mathrm{ind}(m_1,\ldots,m_r;n)\,q^n = \begin{bmatrix} m_1+\cdots+m_r \\ m_1, m_2, \ldots, m_r \end{bmatrix}.$$

<u>Cor.</u> $\mathrm{inv}(m_1,m_2,\ldots,m_r;n) = \mathrm{ind}(m_1,m_2,\ldots,m_r;n)$ for all n.