and $f(m) = 0$ if m is odd. $\square$

(2) We prove (2) by induction on n. Result is clearly true for $n=0$ since $\begin{bmatrix} m \\ m \end{bmatrix} = 1 = \begin{bmatrix} m+1 \\ m+1 \end{bmatrix}$.

Assume result is true for a given n.

$$\begin{bmatrix} n+m+2 \\ m+1 \end{bmatrix} = \begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix} + q^{n+1} \begin{bmatrix} n+m+1 \\ m \end{bmatrix}$$

$$= \sum_{j=0}^{n} q^j \begin{bmatrix} m+j \\ m \end{bmatrix} + q^{n+1} \begin{bmatrix} n+m+1 \\ m \end{bmatrix}$$

$$= \sum_{j=0}^{n+1} q^j \begin{bmatrix} m+j \\ m \end{bmatrix}$$ & result is true for $n+1$, and true in general for $n \geq 0$ by induction.

(3) [q-analog of Chu-Vandermonde summation]

Let $m, n \geq 0$.

$$(z)_{m+n} = (1-z)(1-zq)\cdots(1-zq^{n-1})(1-zq^n)\cdots(1-zq^{n+m-1})$$

(*) $(z)_{m+n} = (z)_n \, (zq^n; q)_m$

$$(z)_{m+n} = \sum_{j=0}^{m+n} \begin{bmatrix} m+n \\ j \end{bmatrix} (-1)^j z^j q^{j(j-1)/2}$$

$$(z)_n (zq^n)_m = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k z^k q^{k(k-1)/2} \cdot \sum_{l=0}^{m} \begin{bmatrix} m \\ l \end{bmatrix} (-1)^l (zq^n)^l q^{l(l-1)/2}$$

$$= \sum_{h=0}^{m+n} (-1)^h z^h q^{h(h-1)/2} \sum_{k=0}^{h} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ h-k \end{bmatrix} q^{(n-k)(h-k)}.$$