and $$(q^{m+1})_r = \frac{(q)_{m+r}}{(q)_m}.$$

Thus
$$\sum_{r=0}^{N} \frac{(q)_{M-m}}{(q)_{M-m-r}} \frac{(q)_{m+n+r}}{(q)_{m+n}} \frac{(q)_N}{(q)_{N-r}} \frac{(q)_m}{(q)_{m+r}} \frac{(q)_{m+n-M-N}}{(q)_{m+n-M-N+r}} \cdot \frac{1}{(q)_r} \cdot q^{(r^2+rm-rM-rN)}$$

$$= \frac{(q)_{M+N}}{(q)_M} \frac{(q)_n}{(q)_{n-N}} \frac{(q)_m}{(q)_{m+N}} \frac{(q)_{n+m-M-N}}{(q)_{n+m-M}} q^{(Nm-NM)}$$

and

$$\sum_{r\geq 0} \left(\frac{(q)_{M-m}}{(q)_{M-m-r}(q)_r}\right)\left(\frac{(q)_{N+m}}{(q)_{m+r}(q)_{N-r}}\right)\left(\frac{(q)_{m+n+r}}{(q)_{M+N}(q)_{m+n+r-M-N}}\right) \cdot q^{(N-r)(M-m-r)}$$

$$= \left(\frac{(q)_{m+n}}{(q)_M (q)_{m+n-M}}\right)\left(\frac{(q)_n}{(q)_N (q)_{n-N}}\right). \quad \square$$