largest part = # of 1s to right of the first 2

second largest part = ... second 2

etc

Thus the # of inversions = sum of parts of the partition.

This gives the desired bijection. The process can be reversed by drawing the boundary from the seq. of 1s & 2s. $\square$

Corollary

$$\sum_{n\geq 0} \mathrm{inv}(m_1, m_2; n) q^n = \begin{bmatrix} m_1+m_2 \\ m_1 \end{bmatrix} = \frac{(q)_{m_1+m_2}}{(q)_{m_1}(q)_{m_2}}.$$

Theorem (P.A. MacMahon)

$$\sum_{n\geq 0} \mathrm{inv}(m_1, m_2, \ldots, m_k; n) q^n = \begin{bmatrix} m_1+m_2+\cdots+m_k \\ m_1, m_2, \ldots, m_k \end{bmatrix}.$$

The Major Index

For a permutation $x_1, x_2, \ldots, x_m$ of the multiset $\{1^{m_1} 2^{m_2} \cdots r^{m_r}\}$ define

$$\chi(x_i) = \begin{cases} i & \text{if } x_i > x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

The greater index of $x_1, x_2, \ldots, x_m = \sum_{i=1}^{m-1} \chi(x_i)$

where $m = m_1 + \cdots + m_r$.