$$\frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-m+1})}{(1-q^m)(1-q^{m-1})\cdots(1-q^1)}\frac{\cancel{(1-q^{n-m})\cdots(1-q)}}{\cancel{(1-q^{n-m})\cdots(1-q)}}$$

Thus $g(N,M;q)$ & $G(N,M;q)$ satisfy the same initial conditions (**) & the same recurrence (***). (4)

By uniqueness,

$$G(N,M;q) = g(N,M;q) = \frac{(q)_{M+N}}{(q)_M (q)_N}. \quad \square$$

GAUSSIAN POLYNOMIALS

The polynomials

$$G(N,M;q) = \sum_{n\geq 0} p(N,M,n) q^n = \frac{(q)_{M+N}}{(q)_M (q)_N}$$

are called <u>Gaussian polynomials</u> or <u>q-binomial coefficients</u>

<u>Definition</u>: The ~~Gauss~~ Gaussian polynomial

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ m \end{bmatrix} := \begin{cases} \dfrac{(q)_n}{(q)_m (q)_{n-m}} & \text{if } 0\leq m\leq n \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$\begin{bmatrix} N \\ M \end{bmatrix} = G(N-M, M; q) \quad \text{if } 0\leq M\leq N.$$

<u>Theorem</u> Let $0\leq m\leq n$ be integers. The Gaussian poly $\begin{bmatrix} n \\ m \end{bmatrix}$ is a poly. (in q) of degree $m(n-m)$ satisfying

(1) $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1,$

(2) $\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ n-m \end{bmatrix}$