Proof

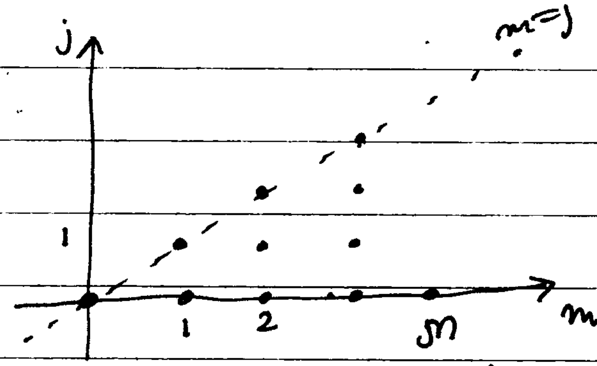
(1) Let $f(m) := \sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix}$ for $m \geq 0$.

Then $$\sum_{m=0}^{\infty} \frac{f(m) z^m}{(q)_m} = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \frac{(-1)^j z^m}{(q)_j (q)_{m-j}}$$ (assume $0<|z|<|q|<1$)

$$= \sum_{j=0}^{\infty} \sum_{m=j}^{\infty} \frac{(-1)^j z^m}{(q)_j (q)_{m-j}}$$ (by abs. conv.)

($m=j+k$)

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^j z^{j+k}}{(q)_j (q)_k}$$



$\{(m,j) : m \geq 0 \ \& \ 0 \leq j \leq m\}$
$= \{(m,j) : j \geq 0 \ \& \ m \geq j\}$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{(q)_j} \sum_{k=0}^{\infty} \frac{z^k}{(q)_k}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{(q)_j} \frac{1}{(z)_\infty}$$ (by Euler's cor. to q-bin thm.)

$$= \frac{1}{(-z)_\infty} \frac{1}{(z)_\infty} = \frac{1}{(z^2;q^2)_\infty} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(q^2;q^2)_k}$$

$$\sum_{m=0}^{\infty} \frac{f(m) z^m}{(q)_m} = \sum_{m=0}^{\infty} \frac{f(2m) z^{2m}}{(q)_{2m}} + \sum_{m=0}^{\infty} \frac{f(2m+1) z^{2m+1}}{(q)_{2m+1}}.$$

Hence

$$\frac{f(2m)}{(q)_{2m}} = \frac{1}{(q^2;q^2)_m}, \quad \& \ f(2m) = \frac{(1-q)(1-q^2)\cdots(1-q^{2m})}{(1-q^2)(1-q^4)\cdots(1-q^{2m})}$$

$$= (1-q)(1-q^3)\cdots(1-q^{2m-1}) = (q;q^2)_m$$