Chapter 5 Identities of the Rogers–Ramanujan Type

(See Ch 7 of TOP)

The Rogers–Ramanujan Identities (Rogers (1894))

(1) $$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

(2) $$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

To be proved later.

It turns out that the LHS is the G.F. for partitions in which difference between parts is at least 2.

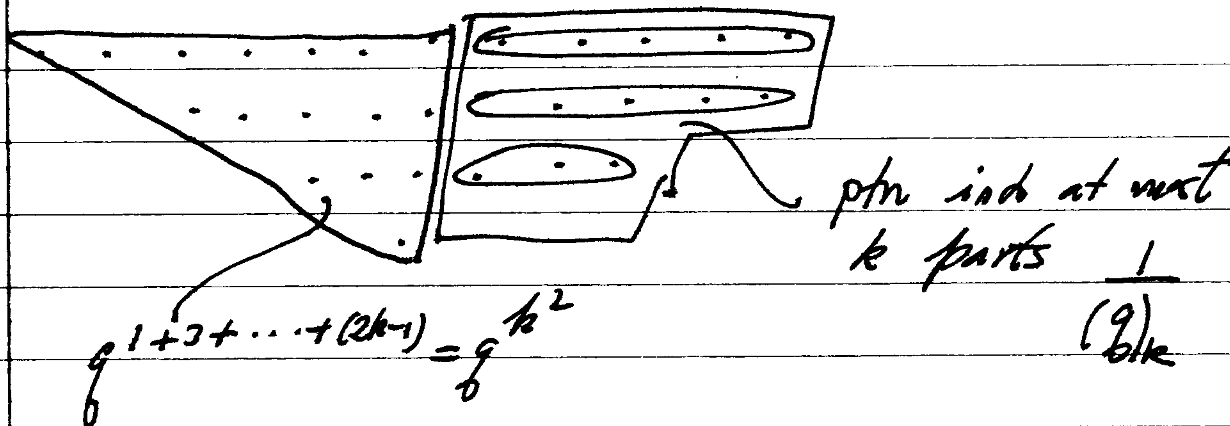
Let $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ be a partition into k parts ~~whose~~ in which difference between parts is at least 2.

So
$$\begin{aligned}
\pi_k &\geq 1\\
\pi_{k-1} &\geq \pi_k + 2 \geq 3\\
\pi_{k-2} &\geq \pi_{k-1} + 2 \geq 5\\
&\vdots\\
\pi_1 &\geq \pi_2 + 2 \geq 2k-1
\end{aligned}$$

If we subtract 1 from π_k, 3 from π_{k-1}, 5 from π_{k-2}, $\ldots$ $(2k-1)$ from π_1 i.e.

We obtain

$$\alpha_k = (\pi_k - 1),\ \alpha_{k-1} = (\pi_{k-1} - 3), \ldots, \alpha_{k-b} = \pi_{k-b} - (2b+1)$$
$$, \ldots, \alpha_1 = (\pi_1 - (2k-1)), \text{ and}$$
$$\alpha_k \geq \alpha_{k-1} \geq \cdots \geq \alpha_2 \geq \alpha_1 = 0.$$



Let $p_k(R, n)$ = # of partitions of n into k parts in which difference between parts ≥ 2.

Let $p(R, n)$ = # of partitions of n in which diff between parts ≥ 2.

Then
$$\sum_{n \geq 0} p_k(R, n) q^n = \frac{q^{k^2}}{(q)_k}$$
$$\sum_{n \geq 0} p(R, n) q^n = \sum_k \left(\sum_{n \geq 0} p_k(R, n) q^n \right) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q)_k}$$

<u>EX</u>: $\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n}$ is G.F. for partitions into parts ≥ 2 in which diff. between parts ≥ 2.

<u>Cor</u>

(1) The number of ptns of n in which diff. between parts ≥ 2 = The number of ptns of n into parts $\equiv 1, 4 \pmod 5$.

(2) The number of ptns of n in which diff. between parts ≥ 2 & no ones = The number of ptns of n into parts $\equiv 2, 3 \pmod 5$.

Chapter 5 Identities of the Rogers-Ramanujan Type

(See Ch 7 of TEXT)

Theorem (Schur, 1926)

Let

$A(n) = \#$ of partitions of n into distinct parts $\equiv 1,2 \pmod 3$.

$B(n) = \#$ of partitions of n into parts $\equiv 1,5 \pmod 6$.

$C(n) = \#$ of partitions of n $\pi = (\pi_1, \pi_2, \dots, \pi_k)$ where

$$\pi_i - \pi_{i+1} \geq 3$$

& $\pi_i - \pi_{i+1} > 3$ if $3 \mid \pi_i$.

Then

$$A(n) = B(n) = C(n)$$

for all $n \geq 1$.

Example ($n=12$)

PTNS enumerated by

$A(12)$	$B(12)$	$C(12)$
11+1	11+1	12
10+2	7+5	11+1
8+4	7+1+1+1+1+1	10+2
7+5	5+5+1+1	9+3
7+4+1	5+1+1+1+1+1+1+1	8+4
5+4+2+1	1+1+... +1	7+4+1

Proof: $\sum_{n=0}^{\infty} A(n) q^n = \prod_{n=0}^{\infty} (1+q^{3n+1})(1+q^{3n+2})$

$$= \prod_{n=0}^{\infty} \frac{(1-q^{6n+2})(1-q^{6n+4})}{(1-q^{3n+1})(1-q^{3n+2})}$$

$$= \frac{\displaystyle\prod_{\substack{n\geq 1 \\ n\equiv 2,4 \pmod 6}} (1-q^n)}{\displaystyle\prod_{\substack{n\geq 1 \\ n\equiv 1,2 \pmod 3}}}$$

$$= \frac{\displaystyle\prod_{\substack{n\geq 1 \\ n\equiv 2,4 \pmod 6}} (1-q^n)}{\displaystyle\prod_{\substack{n\geq 1 \\ n\equiv 1,2,4,5 \pmod 6}} (1-q^n)} = \prod_{\substack{n\geq 1 \\ n\equiv 1,5 \\ \pmod 6}} \frac{1}{1-q^n}$$

$$= \sum_{n=0}^{\infty} B(n) q^n.$$

Hence $A(n) = B(n)$ for all n.

Let $C_m(n) = \#$ of ptns of n enumerated by $C(n)$
($\underline{\text{ie}}$ ptns $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$
where $\pi_i - \pi_{i+1} \geq 3$ &
$\pi_i - \pi_{i+1} > 3$ if $3 \mid \pi_i$)
and the largest part $\pi_1 \leq m$.

$C_m(n) - C_{m-1}(n) = \#$ of ptns of n enumerated by $C(n)$
with largest part $\pi_1 = m$

<u>Case 1.</u> $m \equiv 1$ or $2 \pmod 3$
We remove the largest part and obtain a partition of $n-m$ whose largest part $\leq m-3$ and satisfies the difference conditions.

<u>Case 2.</u> $m \equiv 0 \pmod 3$
We remove the largest part and obtain a partition of $n-m$ whose largest part $\leq m-4$ and satisfies the difference condition.

Hence

$$C_m(n) - C_{m-1}(n) = \begin{cases} C_{m-3}(n-m) & \text{if } m \equiv 1, 2 \pmod 3 \\ C_{m-4}(n-m) & \text{if } m \equiv 0 \pmod 3 \end{cases}$$

So we have

$$C_{3m+1}(n) = C_{3m}(n) + C_{3m-2}(n-3m-1)$$
$$C_{3m+2}(n) = C_{3m+1}(n) + C_{3m-1}(n-3m-2)$$
$$C_{3m+3}(n) = C_{3m+2}(n) + C_{3m-1}(n-3m-3)$$

We let

$$f(q) = \sum_{n=0}^{\infty} C(n) q^n \quad \& \quad f_m(q) = \sum_{n \geq 0} C_m(n) q^n.$$

$$c_{3m+1}(n)q^{n} = c_{3m}(n)q^{n} + q^{3m+1} c_{3m-2}(n-3m-1)\, q^{n-3m-1}$$

$$\sum_{n=0}^{\infty} c_{3m+1}(n)\, q^{n} = \sum_{n=0}^{\infty} c_{3m}(n)q^{n} + q^{3m+1}\sum_{n\geq 3m+1} c_{3m-2}(n-3m-1)\, q^{n-3m-1}$$

$$= \sum_{n=0}^{\infty} c_{3m}(n)q^{n} + q^{3m+1}\sum_{n=0}^{\infty} c_{3m-2}(n)\, q^{n}$$

Hence

$$f_{3m+1}(q) = f_{3m}(q) + q^{3m+1} f_{3m-2}(q).$$

Similarly,

$$f_{3m+2}(q) = f_{3m+1}(q) + q^{3m+2} f_{3m-1}(q)$$

$$f_{3m+3}(q) = f_{3m+2}(q) + q^{3m+3} f_{3m-1}(q).$$

Let $\alpha_m(q) := f_{3m+2}(q)$.

So

$$f_{3m+3}(q) = \alpha_m(q) + q^{3m+3}\alpha_{m-1}(q),$$
$$f_{3m+1}(q) = \alpha_m(q) - q^{3m+2}\alpha_{m-1}(q),$$

and

$$\alpha_m - q^{3m+2}\alpha_{m-1} = \alpha_{m-1} + q^{3m}\alpha_{m-2} + q^{3m+1}(\alpha_{m-1} - q^{3m-1}\alpha_{m-2})$$

Hence,

(*) $\alpha_m = (1 + q^{3m+1} + q^{3m+2})\alpha_{m-1} + q^{3m}(1-q^{3m})\alpha_{m-2}$

for $m \geq 2$. This recurrence together with the initial conditions

$$\alpha_0 = f_2(q) = 1 + q + q^2$$

$$\alpha_1 = f_5(q) = 1 + q + q^2 + q^3 + q^4 + 2q^5 + q^6 + q^7$$

uniquely determine α_n for $n \geq 0$.

<u>Note:</u> We want hold (*) for $n=1$.

$$\alpha_1 = 1 + q + q^2 + q^3 + q^4 + 2q^5 + q^6 + q^7$$
$$= (1 + q^4 + q^5)\alpha_0 + q^3(1-q^3)\alpha_{-1}$$

$$q^3(1-q^3)\alpha_{-1} = 1 + q + q^2 + q^3 + q^4 + 2q^5 + q^6 + q^7$$
$$-(1 + q^4 + q^5)(1 + q + q^2)$$
$$= q^3 - q^6$$

and need $\alpha_{-1} = 1$.

Now, for $|z|<1$ & $|q|<1$ let

$$g(z,q) := \sum_{n=0}^{\infty} \delta_n(q)\, z^n = \frac{(-zq;q^3)_\infty(-zq^2;q^3)_\infty}{(z;q^3)_\infty}$$

Then

$$g(zq^3,q) = \frac{(-zq^4;q^3)_\infty(-zq^5;q^3)_\infty}{(zq^3;q^3)_\infty}$$

$$= g(z,q)\frac{(1-z)}{(1+zq)(1+zq^2)}$$

$$(1-z)g(z,q) = (1 + zq + zq^2 + z^2q^3)\, g(zq^3,q)$$

$$(1-z)\sum_{n=0}^{\infty} \delta_n(q) z^n = (1 + zq + zq^2 + z^2q^3)\sum_{n=0}^{\infty} z^n q^{3n}\delta_n(q)$$

$$\sum_{n=0}^{\infty} s_n z^n - \sum_{n=0}^{\infty} s_n z^{n+1} = \sum_{n=0}^{\infty} s_n q^{3n} z^n$$
$$+ \sum_{n=0}^{\infty} z^{n+1}(q^{3n+1}+q^{3n+2}) s_n$$
$$+ \sum_{n=0}^{\infty} z^{n+2} q^{3n+3} s_n$$

$$\sum_{n=0}^{\infty} s_n z^n - \sum_{n=0}^{\infty} s_{n-1} z^n$$
$$= \sum_{n=0}^{\infty} z^n \left(q^{3n} s_n + (q^{3n-1}+q^{3n-2}) s_{n-1} + q^{3n-3} s_{n-2} \right)$$
(assuming $s_{-1}=s_{-2}=0$).

Hence

$$(**) \quad (1-q^{3n}) s_n = (1+q^{3n-1}+q^{3n-2}) s_{n-1} + q^{3n-3} s_{n-2}$$

for $n \geq 2$. (also true for $n \geq 0$ assuming $s_{-1}=s_{-2}=0$).

$\cancel{\text{Let } S_n = (q^3;q^3)_n s_n}$

$$\cancel{S_{n-1} = (q^3;q^3)_{n-1} s_{n-1} = \frac{(q^3;q^3)_n s_{n-1}}{1-q^{3n}}}$$

$$\cancel{S_{n-2} = (q^3;q^3)_{n-2} s_{n-2} = \frac{(q^3;q^3)_{n-1} s_{n-2}}{1-q^{3n-2}}}$$

We multiply both sides of (**) by $(q^3;q^3)_{n-1}$

$$(q^3;q^3)_n s_n = (1+q^{3n-1}+q^{3n-2})(q^3;q^3)_{n-1} + q^{3n-3}(1-q^{3n-3}) \cdot (q^3;q^3)_{n-2} s_{n-2}$$

Thus

$$S_n = (1+q^{3n-1}+q^{3n-2})S_{n-1} + q^{3n-3}(1-q^{3n-3})S_{n-2}$$

where $S_n = (q^3;q^3)_n \, s_n$,

and

S_{m+1} satisfies the same recurrence (*) as α_m.

But $S_1 = (1-q^3)\, s_1 = (1+q+q^2)s_0 = 1+q+q^2$ (by **)

$S_1 = \alpha_0$

and $S_0 = 1 = \alpha_{-1}$.

It follows that $S_{m+1} = \alpha_m$ for $m \ge -1$.

Hence,

$$s_n = \frac{S_n}{(q^3;q^3)_n} = \frac{\alpha_{n-1}}{(q^3;q^3)_n} \quad \text{for } n \ge 0,$$

and

$$\frac{(-zq;q^3)_\infty(-zq^2;q^3)_\infty}{(z;q^3)_\infty} = \sum_{n=0}^{\infty} \frac{\alpha_{n-1}(q)\, z^n}{(q^3;q^3)_n}.$$

We need the following

Lemma : Suppose $\sum_{n=0}^{\infty} a_n z^n$ converges for $|z|<1$.
Suppose $a_n \ge 0$ for $n \ge 0$ & $\lim_{n\to\infty} a_n = a$.
Then

$$\lim_{z\to 1^-} (1-z)\sum_{n=0}^{\infty} a_n z^n = a.$$

Proof of Lemma: Suppose $0 < x < 1$.
Let $\varepsilon > 0$. Choose N such that

$$|a_n - a| < \varepsilon/2 \quad \text{for } n \geq N.$$

Hence $\quad -\varepsilon/2 < a_n - a < \varepsilon/2$

and $\quad a - \varepsilon/2 < a_n < a + \varepsilon/2$ for $n \geq N$

Therefore,

$$(1-x)\sum_{n=0}^{\infty} a_n x^n = (1-x)\sum_{n=0}^{N} a_n x^n + (1-x)\sum_{n=N+1}^{\infty} a_n x^n$$

$$< (1-x)\left(\sum_{n=0}^{N} a_n\right) + (1-x)(a+\varepsilon/2)\sum_{n=N+1}^{\infty} x^n$$

$$= (1-x)\sum_{n=0}^{N} a_n + (1-x)(a+\varepsilon/2)\frac{x^{N+1}}{1-x}$$

$$< (1-x)\sum_{n=0}^{N} a_n + (a+\varepsilon/2)$$

Hence for

$$0 < 1-x < \frac{\varepsilon}{2}\left(\frac{1}{1+\sum_{n=0}^{N} a_n}\right)$$

$$(1-x)\sum_{n=0}^{\infty} a_n x^n < a + \varepsilon.$$

Similarly it can be shown that $\exists\, \delta > 0$ such that
for $0 < 1-x < \delta$

$$a - \varepsilon < (1-x)\sum_{n=0}^{\infty} a_n x^n.$$

It follows that

$$\lim_{x \to 1^-} (1-x)\sum_{n=0}^{\infty} a_n x^n = a.$$

Now suppose $0 < z < 1$ and $0 < q < 1$. (11)

$$\alpha_m(q) = f_{3m+2}(q) = \sum_{n=0}^{\infty} c_{3m+2}(n) q^n \geq 0$$

and

$$\lim_{m \to \infty} \alpha_m(q) = f(q).$$

Also $\dfrac{\alpha_{n-1}(q)}{(q^3;q^3)_n} \geq 0$ and

$$\lim_{n \to \infty} \frac{\alpha_{n-1}(q)}{(q^3;q^3)_n} = \frac{f(q)}{(q^3;q^3)_\infty}.$$

Hence

$$\frac{f(q)}{(q^3;q^3)_\infty} = \lim_{z \to 1^-} (1-z) \frac{(-zq;q^3)_\infty(-zq^2;q^3)_\infty}{(1-z)(zq^3;q^3)_\infty}$$

$$= \frac{(-q;q^3)_\infty(-q^2;q^3)_\infty}{(q^3;q^3)_\infty}$$

and

$$f(q) = \sum_{n=0}^{\infty} c(n) q^n = (-q;q^3)_\infty(-q^2;q^3)_\infty$$

for $0<q<1$ (and all $|q|<1$ by analytic continuation).

Hence

$$\sum_{n=0}^{\infty} c(n) q^n = \sum_{n=0}^{\infty} A(n) q^n \quad \text{for } |q|<1$$

&

$$c(n) = A(n) \quad \text{for all } n \geq 0. \ \square$$

Proof of the Rogers - Ramanujan Identities

For $m = 0, 1, 2$ let

$$H_m = H_m(z,q) = \sum_{n=0}^{\infty} (-1)^n z^{2n} q^{\frac{1}{2}n(5n+1)-mn} (1 - z^m q^{2mn}) \cdot \frac{1}{(q)_n} \frac{1}{(zq^n;q)_\infty}$$

$$H_{m-1} = \sum_{n=0}^{\infty} (-1)^n z^{2n} q^{\frac{1}{2}n(5n+1)-(m-1)n} \frac{(1 - z^{m-1} q^{2(m-1)n})}{(q)_n (zq^n;q)_\infty}$$

$$H_m - H_{m-1} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} q^{\frac{1}{2}n(5n+1)}}{(q)_n (zq^n;q)_\infty} \cdot \left(q^{-mn} - z^m q^{mn} - q^{-(m-1)n} + z^{m-1} q^{(m-1)n} \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} q^{\frac{1}{2}n(5n+1)}}{(q)_n (zq^n;q)_\infty} \left(z^{m-1} q^{(m-1)n} (1 - zq^n) + q^{mn}(1 - q^n) \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+m-1} q^{\frac{1}{2}n(5n+1)+n(m-1)}}{(q)_n (zq^{n+1};q)_\infty} + \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n} q^{\frac{1}{2}n(5n+1)-mn}}{(q)_{n-1} (zq^n;q)_\infty}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+m-1} q^{\frac{1}{2}n(5n+1)+n(m-1)}}{(q)_n \, (zq^{n+1};q)_\infty}$$

$$+ \sum_{n=0}^{\infty} (-1)^{n+1} \frac{z^{2(n+1)} q^{\frac{1}{2}(n+1)(5n+6)-m(n+1)}}{(q)_n \, (zq^{n+1};q)_\infty}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+m-1} q^{\frac{1}{2}n(5n+1)+n(m-1)} \left(1 - z^{3-m} q^{(2n+1)(3-m)}\right)}{(q)_n \, (zq^{n+1};q)_\infty}$$

since $\frac{1}{2}n(5n+1) + n(m-1) + (2n+1)(3-m)$

$= \frac{1}{2}n(5n+1) + 5n - nm - m + 3$

& $\frac{1}{2}(n+1)(5n+6) = \frac{1}{2}(n+1)((5n+1)+5)$

$= \frac{1}{2}n(5n+1) + \frac{1}{2}(5n + 5n+1 + 5)$

$= \frac{1}{2}n(5n+1) + (5n+3)$

Therefore,

$$H_{3-m}(zq,q) = \sum_{n=0}^{\infty} (-1)^n z^{2n} q^{\frac{1}{2}n(5n+1) - (3-m)n + 2n} \frac{\cdot \left(1 - z^{3-m} q^{2(3-m)n + (3-m)}\right)}{(q)_n \, (zq^n;q)_\infty}$$

Hence

$\cancel{H_{3-m}(zq,q) = z}$

$$H_m(z,q) - H_{m-1}(z,q) = z^{m-1} H_{3-m}(zq,q).$$

$$H_0 = 0$$

$$H_1 - H_0 = H_2(zq)$$

$$H_2 - H_1 = z H_1(zq)$$

Hence

$$H_1(z) = H_2(zq)$$

$$H_2(z) = H_1(z) + z H_1(zq)$$

$$H_2(z) = H_2(zq) + z H_2(zq^2).$$

Let

$$H_2(z) = \sum_{n=0}^{\infty} c_n(q) z^n.$$

Therefore

$$\sum_{n=0}^{\infty} c_n(q) z^n = \sum_{n=0}^{\infty} c_n(q) q^n z^n + \sum_{n=0}^{\infty} c_n(q) q^{2n} z^{n+1}$$

$$\sum_{n=0}^{\infty} c_n(q) z^n = \sum_{n=0}^{\infty} c_n(q) q^n z^n + \sum_{n=1}^{\infty} c_{n-1} q^{2n-2} z^n$$

$$c_0(q) = H_2(0) = 1.$$

For $n \geq 1$,

$$c_n(q) = c_n(q) q^n + q^{2n-2} c_{n-1},$$

$$(1-q^n) c_n(q) = q^{2n-2} c_{n-1}(q)$$

$$c_n = \frac{q^{2n-2}}{1-q^n} c_{n-1}.$$

Iterating we obtain

$$c_n = \frac{q^{2n-2}}{1-q^n} \frac{q^{2n-4}}{1-q^{n-1}} \cdots\cdots \frac{q^0}{1-q^1} c_0$$

and

$$c_n = \frac{q^{n(n-1)}}{(q)_n}.$$

Hence,

$$H_2(z) = \sum_{n=0}^{\infty} \frac{z^n q^{n(n-1)}}{(q)_n}$$

(15)

$$\sum_{n=0}^{\infty} \frac{z^n q^{n(n-1)}}{(q)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} q^{\frac{1}{2}n(5n+1)-2n}(1-z^2q^{4n})}{(q)_n (zq^n;q)_\infty}$$

Letting $z=q$ we obtain

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(5n+1)}(1-q^{4n+2})}{(q)_n (q^{n+1};q)_\infty}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(5n+1)}(1-q^{4n+2})}{(q)_\infty}$$

$$= \frac{1}{(q)_\infty}\left(\sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(5n+1)} + \sum_{n=0}^{\infty} (-1)^{n+1} q^{\frac{n}{2}(5n+1)+4n+2} \right)$$

$m=-1-n$

(ie $n=-1-m$)

$$= \frac{1}{(q)_\infty}\left(\sum_{n=0}^{\infty} (-1)^n q^{\frac{n}{2}(5n+1)} + \sum_{m=-1}^{-\infty} (-1)^m q^{\frac{1}{2}(-1-m)(-5m-4)-4-4m+2} \right)$$

$$= \frac{1}{(q)_\infty}\left(\sum_{n=0}^{\infty} (-1)^n q^{\frac{n}{2}(5n+1)} + \sum_{m=-1}^{-\infty} (-1)^m q^{\frac{1}{2}m(5m+1)} \right)$$

$$= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n}{2}(5n+1)}$$

We need JTP

$$(z)_\infty (q/z)_\infty (q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} z^n$$

Replace q by q^5 and z by q^3:

$$(q^3;q^5)_\infty (q^2;q^5)_\infty (q^5;q^5)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{5n(n-1)/2} q^{3n}$$

$$\prod_{n=0}^{\infty} (1-q^{5n+2})(1-q^{5n+3})(1-q^{5(n+1)}) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n+1)/2}$$

Hence,
$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{\prod_{\substack{n\geq 1 \\ n\equiv 0,2,3 \pmod 5}} (1-q^n)}{\prod_{n\geq 1} (1-q^n)}$$

$$= \prod_{\substack{n\geq 1 \\ n\equiv 1,4 \pmod 5}} \frac{1}{(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}.$$

This completes the proof of the first Rogers Ramanujan identity. The 2nd Rogers-Ramanujan identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$$

can be proved by using
$$H_1(z) = H_2(zq)$$
and letting $z=q$.

(1961) <u>Gordon's Generalization of the Rogers-Ramanujan Identities</u>

Let $k\geq 2$ and $i\geq 1$.

Let $B_{k,i}(n)$ = # of partitions of n of the form $(b_1, b_2, \ldots, b_s)$ where $b_j - b_{j+k-1} \geq 2$ (for each j) and at most $i-1$ of the b_j are equal to 1.

Let $A_{k,i}(n)$ = # of partitions of n into parts $\not\equiv 0, \pm i \pmod{2k+1}$.

Then
$$A_{k,i}(n) = B_{k,i}(n) \quad \text{for all } n.$$

Roger's First Proof of the Rogers–Ramanujan Identities

Define $A_n(\theta)$ by

$$\sum_{n=0}^{\infty} \frac{A_n(\theta)\, z^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-2zq^n\cos\theta + z^2q^{2n})}$$

$$(|q|<1,\ |z|<1).$$

Then

$$(1-2z\cos\theta+z^2)\sum_{n=0}^{\infty}\frac{A_n(\theta)\,z^n}{(q;q)_n} = \sum_{n=0}^{\infty}\frac{A_n(\theta)\,q^n z^n}{(q)_n}$$

$$\sum_{n=0}^{\infty}\frac{A_n(\theta)\,z^n}{(q)_n} - 2\cos\theta\sum_{n=0}^{\infty}\frac{A_n(\theta)\,z^{n+1}}{(q)_n} + \sum_{n=0}^{\infty}\frac{A_n(\theta)\,z^{n+2}}{(q)_n} = \sum_{n=0}^{\infty}\frac{A_n(\theta)\,q^n z^n}{(q)_n}$$

$$\sum_{n=0}^{\infty}\frac{A_n}{(q)_n}z^n - 2\cos\theta\sum_{n=1}^{\infty}\frac{A_{n-1}}{(q)_{n-1}}z^n + \sum_{n=2}^{\infty}\frac{A_{n-2}}{(q)_{n-2}}z^n = \sum_{n=0}^{\infty}\frac{A_n(\theta)\,q^n z^n}{(q)_n}$$

$$A_n\frac{(1-q^n)}{(q)_n} = 2\cos\theta\,\frac{A_{n-1}}{(q)_{n-1}} - \frac{A_{n-2}}{(q)_{n-2}} \qquad \text{for } n \geq 2$$

$$\frac{A_n}{(q)_{n-1}} = 2\cos\theta\,\frac{A_{n-1}}{(q)_{n-1}} - \frac{A_{n-2}}{(q)_{n-2}}$$

Multiplying both sides by $(q)_{n-1}$ we obtain

$$A_n(\theta) = 2\cos\theta\,A_{n-1}(\theta) - (1-q^{n-1})\,A_{n-2}$$

$$\sum_{n=0}^{\infty} \frac{A_n(\theta) z^n}{(q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-2zq^n\cos\theta + z^2 q^{2n})}$$

$$= \prod_{n=0}^{\infty} \frac{1}{(1-z(e^{i\theta}+e^{-i\theta})q^n + z^2 q^{2n})}$$

$$= \prod_{n=0}^{\infty} \frac{1}{(1-ze^{i\theta}q^n)(1-ze^{-i\theta}q^n)}$$

$$= \frac{1}{(ze^{i\theta})_\infty} \frac{1}{(ze^{-i\theta})_\infty}$$

$$= \sum_{s=0}^{\infty} \frac{(ze^{i\theta})^s}{(q)_s} \sum_{t=0}^{\infty} \frac{(ze^{-i\theta})^t}{(q)_t}$$

$$= \sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \frac{e^{i\theta(s-t)} z^{s+t}}{(q)_s (q)_t}.$$

Hence,

$$A_n(\theta) = \sum_{\substack{s+t=n \\ s,t\geq 0}} \frac{(q)_n}{(q)_s (q)_t} e^{i\theta(s-t)}$$

$$= \sum_{\substack{s+t=n \\ s,t\geq 0}} \frac{(q)_n}{(q)_s (q)_t} \cos(s-t)\theta \qquad \text{(by taking real part).}$$

$$= \sum_{t=0}^{n} \frac{(q)_n}{(q)_{n-t} (q)_t} \cos(n-2t)\theta$$

$$= \sum_{t=0}^{n} \begin{bmatrix} n \\ t \end{bmatrix} \cos(n-2t)\theta = 2{\sum_{0\le 2t\le n}}' \begin{bmatrix} n \\ t \end{bmatrix} \cos(n-2t)$$

Problem: Suppose (where $\sum'$ means that if n is even the term with $t=\frac{n}{2}$ is multiply $1/2$)

$$\sum_{n=0}^{\infty} a_n A_n(\theta) = b_0 + \sum_{n=1}^{\infty} 2 b_n \cos n\theta \qquad (\text{for } \theta\in\mathbb{R})$$

Find identities for the a_n's in terms of the b_n's & vice versa.

$$\sum_{n=0}^{\infty} a_n {\sum_{0\le 2t\le n}}' 2 \begin{bmatrix} n \\ t \end{bmatrix} \cos(n-2t) = b_0 + \sum_{n=1}^{\infty} 2b_n \cos n\theta$$

$$b_0 = \sum_{n=0}^{\infty} \begin{bmatrix} 2n \\ n \end{bmatrix} a_{2n}$$

and

$$(*) \qquad b_r = \sum_{t\ge 0} \begin{bmatrix} r+2t \\ t \end{bmatrix} a_{r+2t}$$

$(n-2t=r,\ n=r+2t)$

Suppose we formally invert $(*)$:

$$(**) \qquad a_r = \sum_{h\ge 0} T(r,h)\, b_{r+h}$$

We substitute $(**)$ into $(*)$:

$$b_r = \sum_{t\geq 0} \begin{bmatrix} r+2t \\ t \end{bmatrix} a_{r+2t}$$

$$= \sum_{t\geq 0} \begin{bmatrix} r+2t \\ t \end{bmatrix} \sum_{h\geq 0} T(r+2t,h)\, b_{r+2t+h}$$

$$= \sum_{t,h\geq 0} \begin{bmatrix} r+2t \\ t \end{bmatrix} T(r+2t,h)\, b_{r+2t+h}$$

We require

$$T(r,0)=1.$$

Let $N = 2t+h > 0$ so that $h = N-2t$.

We require

$$(**)' \quad \sum_{0\leq 2t\leq N} \begin{bmatrix} r+2t \\ t \end{bmatrix} T(r+2t, N-2t) = 0$$

for $N>0$. We want

$$(***) \begin{cases} T(r,0)=1, \\ T(r,1)=0, \qquad \text{(by letting } N=1) \\ \text{and} \\ T(r,N) = -\sum_{1\leq 2j\leq N} \begin{bmatrix} r+2j \\ j \end{bmatrix} T(r+2j, N-2j). \end{cases}$$

$(***)$ uniquely define $T(r,n)$ for all $r, n\geq 0$.

It is clear that

$$T(r,N) = 0 \qquad \text{if } N \text{ is odd.}$$

It seems that

<u>Theorem</u>: If $T(r,n)$ satisfies $(**)'$ Then we have:

(1) $T(r, 2n+1) = 0$

(2) $T(r, 2n) = (-1)^n q^{n(n-1)/2} \dfrac{(q^{r+1})_{n-1}(1-q^{r+2n})}{(q)_n}$

<u>Proof</u>:

From p.11 of Notes Ch. 4,

$$\sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j (q)_j} t^j = \frac{(c/b)_\infty (bt)_\infty}{(c)_\infty (t)_\infty} \sum_{m=0}^{\infty} \frac{(\frac{at}{c} b)_m (b)_m (c/b)^m}{(bt)_m (q)_m}$$

(by Heine's transf).

Let $b = q^{-n}$.

$$\frac{(c/b)_\infty}{(c)_\infty} = \frac{(q^n c)_\infty}{(c)_\infty} = \frac{1}{(c)_n}$$

$$\frac{(bt)_\infty}{(t)_\infty} = \frac{(q^{-n}t)_\infty}{(t)_\infty} = (1-q^{-n}t)(1-q^{-n+1}t)\cdots(1-q^{-1}t)$$

$$= \cancel{(-1)^n q^{-n+(-n+1)+\cdots+(-1)} t^n (t^{-1}q)_n}$$

$$= (tq^{-n})_n .$$

Hence

$$\sum_{j=0}^{\infty} \frac{(a)_j (q^{-n})_j}{(c)_j (q)_j} t^j = \frac{(tq^{-n})_n}{(c)_n} \sum_{j\geq 0} \frac{(q^{-n})_j (atq^{-n}/c)_j (cq^n)^j}{(q^{-n}t)_j (q)_j}$$

$$\sum_{j=0}^{n} \frac{(a)_j (q^{-n})_j t^j}{(c)_j (q)_j} = \frac{1}{(c)_n} \sum_{j=0}^{n} \frac{(atq^{-n}/c)_j}{(q)_j} (tq^{-n+j})_{n-j} (q^{-n})_j \cdot (cq^n)^j$$

(since $(q^{-n})_j = 0$ if $j > n$)

$$\frac{(tq^{-n})_n}{(tq^{-n})_j} = \begin{cases} 1 & \text{if } j=n \\ (1-tq^{-n+j})\cdots(1-tq^{-1}) & \text{if } 0\le j<n \end{cases}$$

Letting $t \to q$ we find that

$$\sum_{j=0}^{n} \frac{(a)_j (q^{-n})_j q^j}{(c)_j (q)_j} = \frac{(q^{-n})_n (aq^{1-n}/c)_n}{(c)_n (q)_n} c^n q^{n^2}$$

$$= \frac{(q)_n \, (c/a)_n \, c^n q^{n^2}}{(c)_n (q)_n \, (-q)^n q^{n(n-1)/2} (-c/a)^n q^{n(n-1)/2}}$$

$$\boxed{\sum_{j=0}^{n} \frac{(a)_j (q^{-n})_j q^j}{(c)_j (q)_j} = \frac{a^n \cancel{(q)_n} (c/a)_n}{(c)_n \cancel{(q)_n}}}$$

Let $\tau(r, 2n+1) = 0$ for $n \geq 0$

and $\tau(r, 2n) = (-1)^n q^{n(n-1)/2} \dfrac{(q^{r+1})_{n-1}(1-q^{r+2n})}{(q)_n}$, for $n \geq 1$

& $\tau(r,0) = 1$.

We must show $\tau(r,n)$ satisfies the same recurrence as $T(r,n)$ (***) ie (**)', which clearly holds when N is odd.

Let $N = 2n > 0$.

$$\sum_{0\le 2j\le 2n} \begin{bmatrix} r+2j \\ j \end{bmatrix} \tau(r+2j,\ 2n-2j)$$

$$= \sum_{j=0}^{n} \begin{bmatrix} r+2j \\ j \end{bmatrix} \frac{(-1)^{n-j} q^{(n-j)(n-j-1)/2} (q^{r+2j+1})_{n-j-1} (1-q^{r+2n})}{(q)_{n-j}}$$

$$= \sum_{j=0}^{n} \frac{(-1)^{n-j} q^{(n-j)(n-j-1)/2} (q)_{r+2j} (q^{r+2j+1})_{n-j-1} (1-q^{r+2n})}{(q)_j (q)_{r+j} (q)_{n-j}}$$

$$= \sum_{j=0}^{n} \frac{(-1)^{n-j} q^{(n-j)(n-j-1)/2} (q)_{r+j+n-1} (1-q^{r+2n})}{(q)_j (q)_{r+j} (q)_{n-j}}$$

[Recall $(q^{m+1})_j (q)_m = (q)_{m+j}$

and

$$(q^{-n})_j = \frac{(q)_n}{(q)_{n-j}} (-1)^j q^{j(j-1)/2 - nj}$$

$$\& \quad \frac{1}{(q)_{n-j}} = \frac{(q^{-n})_j}{(q)_n \ (-1)^j q^{j(j-1)/2 - nj}} \quad]$$

$$= \sum_{j=0}^{n} \frac{(-1)^n (1-q^{r+2n}) q^{n(n-1)/2+j} (q^{r+n})_j (q^{-n})_j (q)_{r+n-1}}{(q)_j (q^{r+1})_j (q)_r (q)_n}$$

$$= \frac{(-1)^n (1-q^{r+2n}) q^{n(n-1)/2} (q)_{r+n-1}}{(q)_r (q)_n} \sum_{j=0}^{n} \frac{(q^{-n})_j (q^{r+n})_j q^j}{(q^{r+1})_j (q)_j}$$

$$= \frac{(-1)^n (1-q^{r+2n}) q^{n(n-1)/2} (q)_{r+n+1}}{(q)_r (q)_n} \frac{(q^{r+n})^n (q^{1-n})_n}{(q^{r+1})_n}$$

$(c = q^{r+1},\ a = q^{r+n}$ in q-Chu-V.)

$= 0$ if $n > 0$ since $(q^{1-n})_n = (1-q^{1-n}) \cdots (1-q^0) = 0.$

Theorem: If for each $r \geq 0$,

$$b_r = \sum_{j \geq 0} \begin{bmatrix} r+2j \\ j \end{bmatrix} a_{r+2j}$$

then

$$a_r = \sum_{h \geq 0} \frac{(-1)^h q^{h(h-1)/2} (q^{r+1}; q)_h (1-q^{r+2h})}{(q)_h (1-q^{r+h})} b_{r+2h}$$

(assuming convergence conditions).

Let $F(\lambda, \theta) = \prod_{n=1}^{\infty} (1 + 2\lambda q^n \cos\theta + \lambda^2 q^{2n})$

We find $\{a_r\}, \{b_r\}$ such that

$$F(\lambda, \theta) = \sum_{r=0}^{\infty} a_r A_r(\theta)$$

and

$$F(\lambda, \theta) = b_0 + \sum_{r=1}^{\infty} 2 b_r \cos r\theta .$$

$F(\lambda,\theta)=$ (27)

$$\prod_{n=1}^{\infty}(1+2\lambda q^n\cos\theta+\lambda^2q^{2n}) = (-\lambda e^{i\theta}q;q)_\infty(-\lambda e^{-i\theta}q;q)_\infty$$

$$= \sum_{r=0}^{\infty}\frac{(\lambda e^{i\theta})^r q^{r(r+1)/2}}{(q)_r}\sum_{s=0}^{\infty}\frac{(+\lambda e^{-i\theta})^s q^{s(s+1)/2}}{(q)_s}$$

$$= \sum_{r,s\geq 0}\frac{\lambda^{r+s}e^{i\theta(r-s)}q^{(r+s+1)(r+s)/2}}{(q)_r(q)_s}\cdot q^{-rs}$$

Using q-Chu-Van:

$$\sum_{j\geq 0}\frac{(a)_j(q^{-s})_j q^j}{(c)_j(q)_j} = \frac{a^s(c/a)_s}{(c)_s}$$

with $c=0$ & $a=q^{-r}$:

$$\sum_{j\geq 0}\frac{(+q^{-r})_j(q^{-s})_j q^j}{(q)_j} = q^{-rs}.$$

Hence,

$$F(\lambda,\theta)=\sum_{r,s\geq 0}\frac{\lambda^{r+s}e^{i\theta(r-s)}q^{(r+s+1)(r+s)/2}}{(q)_r\,(q)_s}\sum_{j\geq 0}\frac{(q^{-r})_j(q^{-s})_jq^j}{(q)_j}$$

$$= \sum_{r,s\geq 0}\sum_{j=0}^{\min(r,s)}\frac{\lambda^{r+s}e^{i\theta(r-s)}q^{(r+s+1)(r+s)/2}q^j q^{j(j-1)-(r+s)j}}{(q)_{r-j}(q)_{s-j}(q)_j}$$

$(r\to r+j,\ s\to s+j)$

$$= \sum_{j=0}^{\infty} \sum_{r,s\geq 0} \frac{\lambda^{r+s+2j} e^{i\theta(r-s)} q^{(r+s+j)(r+s+j-1)/2}}{(q)_r (q)_s (q)_j} \cdot q^{(j+1)j/2+r+s+j} \quad (28)$$

$$= \sum_{j,n\geq 0} \sum_{\substack{r,s\geq 0\\ r+s=n}} \left(\quad \right)$$

$$= \sum_{j,n\geq 0} \frac{\lambda^{n+2j} q^{(n+j)(n+j-1)/2+(j+1)j/2+n+j}}{(q)_n \cancel{(q)_r(q)_s} (q)_j} \cdot \sum_{\substack{r,s\geq 0\\ r+s=n}} \frac{(q)_n}{(q)_r (q)_s} e^{i\theta(r-s)}$$

$$= \sum_{n=0}^{\infty} \frac{\lambda^n}{(q)_n} q^{n(n+1)/2} A_n(\theta) \sum_{j=0}^{\infty} \frac{\lambda^{2j} q^{j^2+(n+1)j}}{(q)_j}$$

We need JTP

$$(z)_\infty (q/z)_\infty (q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n(n-1)/2}$$

$$F\left(\frac{1}{\sqrt{q}}, \theta\right) = (-\sqrt{q}\, e^{i\theta}; q)_\infty (-\sqrt{q}\, e^{-i\theta}; q)$$

$$= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (\sqrt{q})^n e^{ni\theta} q^{n(n-1)/2} \qquad (z = -\sqrt{q}\, e^{i\theta})$$

$$= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} q^{n^2/2} e^{in\theta} = \frac{1}{(q)_\infty} \left(1 + 2\sum_{n=1}^{\infty} \cos(n\theta)\, q^{n^2/2}\right)$$

Hence,

$$a_r = \frac{q^{r^2/2}}{(q)_r} \cdot \sum_{j=0}^{\infty} \frac{q^{j^2+(r+1)j - j}}{(q)_j}$$

and

$$b_0 = \frac{1}{(q)_\infty}$$

$$b_r = \frac{q^{r^2/2}}{(q)_\infty} \quad \text{for } r \geqslant 1.$$

So $\quad b_r = \dfrac{q^{r^2/2}}{(q)_\infty} \quad$ for $r \geqslant 0$.

Now $\quad a_0 = \displaystyle\sum_{j=0}^{\infty} \frac{q^{j^2}}{(q)_j}.$

$$a_0 = \sum_{h \geqslant 0} T(0, h)\, b_h = \sum_{h \geqslant 0} T(0, 2h)\, b_{2h}$$

Hence,

$$\sum_{j=0}^{\infty} \frac{q^{j^2}}{(q)_j} = \frac{1}{(q)_\infty} + \sum_{h \geqslant 1} (-1)^h q^{h(h-1)/2} \frac{(q)_h}{(q)_h} \frac{(1-q^{2h})}{(1-q^h)} \cdot \frac{q^{2h^2}}{(q)_\infty}$$

$$= \frac{1}{(q)_\infty}\left(1 + \sum_{h \geqslant 1} (-1)^h q^{h(5h-1)/2} (1+q^h)\right)$$

$$= \frac{1}{(q)_\infty}\left(1 + \sum_{h \geqslant 1} (-1)^h q^{h(5h-1)/2} + \sum_{h \geqslant 1} (-1)^h q^{h(5h+1)/2}\right)$$

$$= \frac{1}{(q)_\infty}\left(1 + \sum_{h \geq 1} (-1)^h q^{h(5h-1)/2} + \sum_{h=-1}^{-\infty} (-1)^h q^{h(5h-1)/2}\right)$$

$$= \frac{1}{(q)_\infty} \sum_{h=-\infty}^{\infty} (-1)^h q^{h(5h-1)/2}$$

$$= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n+1)/2} \qquad (h=-n)$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})} \qquad \text{(as before).}$$

<u>Bailey's Transform</u> (subject to suitable convergence conditions) if

$$\beta_n = \sum_{r=0}^{n} \alpha_r \, u_{n-r} \, v_{n+r}$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r \, u_{r-n} \, v_{r+n}$$

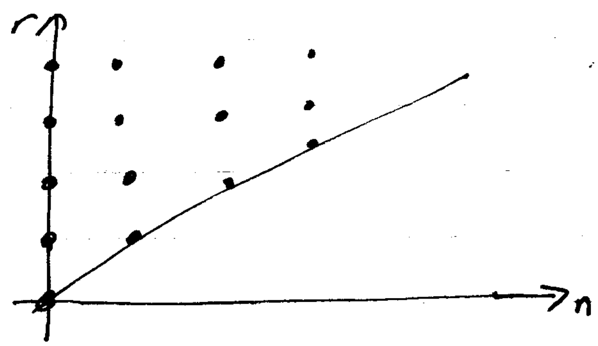
then

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n .$$

<u>Proof:</u>

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \alpha_n \sum_{r=n}^{\infty} \delta_r \, u_{r-n} \, v_{r+n}$$

$$= \sum_{n=0}^{\infty} \sum_{r=n}^{\infty} \alpha_n \, \delta_r \, u_{r-n} \, v_{r+n}$$



$$= \sum_{r=0}^{\infty} \sum_{n=0}^{r} \alpha_n \, \delta_r \, u_{r-n} \, v_{r+n}$$

$$= \sum_{r=0}^{\infty} \delta_r \sum_{n=0}^{r} \alpha_n \, u_{r-n} \, v_{r+n}$$

$$= \sum_{r=0}^{\infty} \delta_r \beta_r = \sum_{n=0}^{\infty} \delta_n \beta_n . \qquad \square$$

Bailey's Lemma If for $n \geq 0$

$$(*) \qquad \beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}\,(aq)_{n+r}}$$

then

$$\beta_n' = \sum_{r=0}^{n} \frac{\alpha_r'}{(q)_{n-r}\,(aq)_{n+r}}$$

where
$$\alpha_r' = \frac{(\rho_1)_r (\rho_2)_r (aq/\rho_1\rho_2)^r \alpha_r}{(aq/\rho_1)_r\,(aq/\rho_2)_r}$$

&

$$\beta_n' = \sum_{j=0}^{n} \frac{(\rho_1)_j (\rho_2)_j (aq/\rho_1\rho_2)_{n-j} \left(\frac{aq}{\rho_1\rho_2}\right)^j \beta_j}{(q)_{n-j}\,\left(\frac{aq}{\rho_1}\right)_n \left(\frac{aq}{\rho_2}\right)_n}$$

Note: A pair of sequences (α_n, β_n) related by $(*)$ is called a Bailey pair. So (α_n', β_n') is a new Bailey pair.

Proof: We apply the Bailey transform with
$$\delta_n = \frac{(\rho_1)_n (\rho_2)_n (q^{-N})_n\, q^n}{(\rho_1\rho_2 q^{-N}/a)_n}$$

$$u_n = \frac{1}{(q)_n} \qquad v_n = \frac{1}{(aq)_n}.$$

We need q-Pfaff-Saalschütz:

$$\sum_{j=0}^{N} \frac{(a)_j (b)_j (q^{-N})_j\, q^j}{(c)_j (abq^{1-N}/c)_j (q)_j} = \frac{(c/a)_N (c/b)_N}{(c)_N (c/ab)_N}.$$

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r \, u_{r-n} \, v_{r+n}$$

$$= \sum_{r=n}^{\infty} \frac{(\rho_1)_r (\rho_2)_r (q^{-N})_r \, q^r}{(\rho_1\rho_2 q^{-N}/a)_r \, (q)_{r-n} \, (aq)_{n+r}}$$

$$= \sum_{r=0}^{\infty} \frac{(\rho_1)_{r+n} (\rho_2)_{r+n} (q^{-N})_{r+n} \, q^{r+n}}{(\rho_1\rho_2 \frac{q^{-N}}{a})_{r+n} \, (q)_r \, (aq)_{r+2n}}$$

$$\left(\text{now} \quad (a)_{r+n} = (a)_r (q^r a)_n = (a)_n (q^n a)_r\right)$$

$$= \frac{(\rho_1)_n (\rho_2)_n (q^{-N})_n \, q^n}{(\rho_1\rho_2 q^{-N}/a)_n \, (aq)_{2n}} \sum_{r=0}^{\infty} \frac{(q^n \rho_1)_r (q^n \rho_2)_r (q^{-(N-n)})_r \, q^r}{(\rho_1\rho_2 q^{-N+n}/a)_r \, (a q^{2n+1})_r \, (q)_r}$$

$$= \frac{(\rho_1)_n (\rho_2)_n (q^{-N})_n \, q^n}{(\rho_1\rho_2 q^{-N}/a)_n \, (aq)_{2n}} \; \frac{(a q^{n+1}/\rho_1)_{N-n} \, (a q^{n+1}/\rho_2)_{N-n}}{(a q^{2n+1})_{N-n} \, (\frac{aq}{\rho_1\rho_2})_{N-n}}$$

$$\left(\text{Recall,} \quad \frac{(x)_{N-n}}{(x)_N} = \frac{1}{(q^{1-N}/x)_n} \; \frac{1}{(-x q^{N-n})^n \, q^{n(n-1)/2}}\right.$$

&

$$(x q^{n+1})_{N-n} = (1 - x q^{n+1}) \cdots (1 - x q^N)$$

$$\left. = \frac{(xq)_N}{(xq)_n}\right)$$

$$= \frac{(aq/\rho_1)_N}{(aq/\rho_1)_n} \; \frac{(aq/\rho_2)_N}{(aq/\rho_2)_n} \; \frac{(\rho_1)_n (\rho_2)_n (q^{-N})_n \left(-\frac{a q^{1+N-n}}{\rho_1\rho_2}\right)^n q^{n(n-1)/2}}{(aq/(\rho_1\rho_2))_N \; \cancel{(\frac{aq}{\rho_1\rho_2})_N}}$$

$$\cdot \frac{q^n}{(aq)_{N+n}}$$

$$(aq)_{N+n} = (aq)_N \, (aq^{N+1})_n$$

so

$$\gamma_n = \frac{(aq/\rho_1)_N (aq/\rho_2)_N}{(aq)_N (aq/(\rho_1\rho_2))_N} (-1)^n \frac{(\rho_1)_n(\rho_2)_n(q^{-N})_n \left(\frac{aq}{\rho_1\rho_2}\right)^n q^{nN-n(n-1)/2}}{(aq/\rho_1)_n (aq/\rho_2)_n (aq^{N+1})_n}$$

Now,

$$\sum_{r=0}^{N} \frac{\alpha_r'}{(q)_{N-r}(aq)_{N+r}}$$

$$= \sum_{r=0}^{N} \frac{(\rho_1)_r(\rho_2)_r(aq/\rho_1\rho_2)^r \alpha_r}{(aq/\rho_1)_r(aq/\rho_2)_r (q)_{N-r}(aq)_{N+r}}$$

$$\left((q)_{N-r} = \frac{(q)_N}{(q^{-N})_r (-q^{1+N-r})^r \, q^{r(r-1)/2}} \right)$$

$$= \sum_{r=0}^{N} \frac{(\rho_1)_r(\rho_2)_r(q^{-N})_r (aq/\rho_1\rho_2)^r (-1)^r}{(aq/\rho_1)_r (aq/\rho_2)_r (aq)_{N+r} (q)_N} q^{rN-r(r-1)/2} \alpha_r$$

$$= \frac{(aq/\rho_1\rho_2)_N}{(aq/\rho_1)_N (aq/\rho_2)_N (q)_N} \sum_{r=0}^{N} \gamma_r \alpha_r$$

$$= \frac{(aq/\rho_1\rho_2)_N}{(aq/\rho_1)_N(aq/\rho_2)_N} \frac{1}{(q)_N} \sum_{r=0}^{N} \beta_r \delta_r \qquad \text{(by Bailey Transf.)}$$

$$= \frac{(aq/\rho_1\rho_2)_N}{(aq/\rho_1)_N(aq/\rho_2)_N} \frac{1}{(q)_N} \sum_{r=0}^{N} \frac{(\rho_1)_r(\rho_2)_r(q^{-N})_r q^r}{(\rho_1\rho_2 q^{-N}/a)_r} \beta_r$$

$$\frac{(q^{-N})_r}{(q)_N} = \frac{1}{(q)_{N-r}} \frac{1}{(-q^{1+N-r})^r q^{r(r-1)/2}}$$

$$\frac{(x)_N}{(q^{1-N}/x)_r} = (x)_{N-r} (-x q^{N-r})^r q^{r(r-1)/2}$$

Hence

$$\sum_{r=0}^{N} \frac{\alpha_r'}{(q)_{N-r} (aq)_{N+r}}$$

$$= \sum_{r=0}^{N} \frac{(\rho_1)_r (\rho_2)_r}{(q)_{N-r}} \frac{(aq/\rho_1\rho_2)_{N-r}}{(aq/\rho_1)_N (aq/\rho_2)_N} \left(\frac{aq}{\rho_1\rho_2}\right)^r \beta_r$$

$$= \beta_N'. \qquad \Box$$

<u>Theorem</u> If (α_n, β_n) is a Bailey pair then

$$\sum_{j\geq 0} a^j q^{j^2} \beta_j = \frac{1}{(aq)_\infty} \sum_{r=0}^{\infty} a^r q^{r^2} \alpha_r.$$

<u>Proof</u> $(\rho_1)_r \frac{1}{\rho_1^r} = (1-\rho_1)(1-\rho_1 q) \cdots (1-\rho_1 q^{r-1}) \frac{1}{\rho_1^r}$

$= (\frac{1}{\rho_1} - 1)(\frac{1}{\rho_1} - q) \cdots (\frac{1}{\rho_1} - q^{r-1})$

$$\lim_{\rho_1 \to \infty} (\rho_1)_r \rho_1^{-r} = (-1)^r q^{r(r-1)/2}$$

$$\lim_{\rho_1, \rho_2 \to \infty} \alpha_r' = a^r q^{r^2} \alpha_r$$

$$\lim_{\rho_1,\rho_2\to\infty} \beta_n' = \sum_{j=0}^{n} \frac{a^j q^{j^2}}{(q)_{n-j}} \beta_j$$

Hence by Bailey's lemma,

$$\sum_{j=0}^{n} \frac{a^j q^{j^2}}{(q)_{n-j}} \beta_j = \sum_{r=0}^{n} \frac{a^r q^{r^2} \alpha_r}{(q)_{n-r}(aq)_{n+r}}$$

Letting $n\to\infty$ (& assuming certain convergence conditions)

$$\frac{1}{(q)_\infty} \sum_{j=0}^{\infty} a^j q^{j^2} \beta_j = \sum_{r=0}^{\infty} \frac{a^r q^{r^2} \alpha_r}{(q)_\infty (aq)_\infty}$$

and

$$\sum_{j=0}^{\infty} a^j q^{j^2} \beta_j = \frac{1}{(aq)_\infty} \sum_{r=0}^{\infty} a^r q^{r^2} \alpha_r .$$

<u>**Cor**</u> If (α_n, β_n) is a Bailey pair then

(α_n', β_n') is a Bailey pair

where

$$\alpha_n' = a^n q^{n^2} \alpha_n$$

$$\beta_n' = \sum_{j=0}^{n} \frac{a^j q^{j^2} \beta_j}{(q)_{n-j}} .$$

<u>Theorem</u>

$$\alpha_n = \begin{cases} 1 & \text{if } n=0 \\ (-1)^n q^{n(n-1)/2}(1+q^n) & \text{if } n \geq 1 \end{cases}$$

$$\beta_n = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n\geq 1 \end{cases}$$

is a Bailey pair with $a=1$.

<u>Proof</u>: We need

$$(z)_N = \sum_{j=0}^{N} (-1)^j z^j q^{j(j-1)/2} \begin{bmatrix} N \\ j \end{bmatrix}$$

(See p. 36 of TEXT or p. 6 of Notes for Ch. 4).
We have to show that

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(q)_{n+r}}.$$

Clearly true when $n=0$ since $\beta_0=\alpha_0=1$.
Let $n\geq 1$. Then

$$\sum_{j=0}^{n} \frac{\alpha_j}{(q)_{n-j}(q)_{n+j}} = \frac{1}{(q)_n^2} + \sum_{j=1}^{n} \frac{(-1)^j q^{j(j-1)/2}(1+q^j)}{(q)_{n-j}(q)_{n+j}}$$

$$= \sum_{j=-n}^{n} \frac{(-1)^j q^{j(j-1)/2}}{(q)_{n-j}(q)_{n+j}}$$

$(r = n+j, \quad j = r-n)$

$$= \sum_{r=0}^{2n} \frac{(-1)^{r-n} q^{(r-n)(r-n-1)/2}}{(q)_r \, (q)_{2n-r}}$$

$$= \frac{(-1)^n}{(q)_{2n}} \sum_{r=0}^{2n} (-1)^r q^{r(r-1)/2 - rn + n(n+1)/2} \begin{bmatrix} 2n \\ r \end{bmatrix}$$

$$= \frac{(-1)^n q^{n(n+1)/2}}{(q)_{2n}} (q^{-n})_{2n} = 0$$

$\cancel{= \begin{cases} 1 & \text{if } n=0 \end{cases}}$

since $n \geq 1$ & $(q^{-n})_{2n} = (1-q^{-n}) \cdots (1-q^{0}) \cdots (1-q^{n-1}) = 0.$

$\square$

Hence

$$\alpha_n' = q^{n^2} \alpha_n = \begin{cases} 1 & \text{if } n=0 \\ (-1)^n q^{n(3n-1)/2}(1+q^n), & n \geq 1 \end{cases}$$

$$\beta_n' = \sum_{j=0}^{n} \frac{a^j q^{j^2} \beta_j}{(q)_{n-j}} = \frac{1}{(q)_n}$$

form a Bailey pair.

Therefore, $$\sum_{j=0}^{\infty} q^{j^2}\beta_j' = \frac{1}{(q)_\infty}\sum_{n=0}^{\infty} q^{n^2}\alpha_n'. \quad (40)$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n} = \frac{1}{(q)_\infty}\left(1+\sum_{n\geq 1}(-1)^n q^{n(5n-1)/2}(1+q^n)\right)$$

(as before).

$$= \prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

Iterating Further

$$\alpha_n'' = q^{n^2}\alpha_n' = \begin{cases} 1 & \text{if } n=0 \\ (-1)^n q^{n(5n-1)/2}(1+q^n) & \text{if } n\geq 1 \end{cases}$$

$$\beta_n'' = \sum_{j=0}^{n}\frac{q^{j^2}}{(q)_{n-j}}\beta_j' = \sum_{j=0}^{n}\frac{q^{j^2}}{(q)_{n-j}(q)_j}$$

forms a Bailey pair.

Hence $$\sum_{n=0}^{\infty} q^{n^2}\beta_n'' = \frac{1}{(q)_\infty}\sum_{n=0}^{\infty} q^{n^2}\alpha_n''$$

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{q^{n^2+j^2}}{(q)_{n-j}(q)_j} = \frac{1}{(q)_\infty}\left(1+\sum_{n\geq 1}(-1)^n q^{n(7n-1)/2}(1+q^n)\right).$$

$$= \prod_{\substack{n\geq 1 \\ n\not\equiv 0,\pm 3 \pmod 7}}\frac{1}{(1-q^n)}$$