Chapter 5 Identities of the Rogers–Ramanujan Type

(See Ch 7 of TOP)

The Rogers–Ramanujan Identities (Rogers (1894))

$$(1) \quad \sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

$$(2) \quad \sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

To be proved later.
It turns out that the LHS is the G.F. for partitions in which difference between parts is at least 2.
Let $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ be a partition into k parts ~~where~~ in which difference between parts is at least 2.
So

$$\begin{aligned} \pi_k &\geq 1 \\ \pi_{k-1} &\geq \pi_k + 2 \geq 3 \\ \pi_{k-2} &\geq \pi_{k-1}+2 \geq 5 \\ &\vdots \\ \pi_1 &\geq \pi_2 + 2 \geq 2k-1 \end{aligned}$$

If we subtract 1 from π_k, 3 from π_{k-1}, 5 from π_{k-2}, ... $(2k-1)$ from π_1 ie.