Proof of Lemma: Suppose $0 < x < 1$.
Let $\varepsilon > 0$. Choose N such that

$$|a_n - a| < \varepsilon/2 \quad \text{for } n \geq N.$$

Hence $-\varepsilon/2 < a_n - a < \varepsilon/2$
and $a - \varepsilon/2 < a_n < a + \varepsilon/2$ for $n \geq N$
Therefore,

$$(1-x)\sum_{n=0}^{\infty} a_n x^n = (1-x)\sum_{n=0}^{N} a_n x^n + (1-x)\sum_{n=N+1}^{\infty} a_n x^n$$

$$< (1-x)\left(\sum_{n=0}^{N} a_n\right) + (1-x)(a+\varepsilon/2)\sum_{n=N+1}^{\infty} x^n$$

$$= (1-x)\sum_{n=0}^{N} a_n + (1-x)(a+\varepsilon/2)\frac{x^{N+1}}{1-x}$$

$$< (1-x)\sum_{n=0}^{N} a_n + (a+\varepsilon/2)$$

Hence for

$$0 < 1-x < \frac{\varepsilon}{2}\left(\frac{1}{1+\sum_{n=0}^{N} a_n}\right)$$

$$(1-x)\sum_{n=0}^{\infty} a_n x^n < a + \varepsilon.$$

Similarly it can be shown that $\exists\, \delta > 0$ such that
for $0 < 1-x < \delta$

$$a - \varepsilon < (1-x)\sum_{n=0}^{\infty} a_n x^n.$$

It follows that

$$\lim_{x \to 1^-} (1-x)\sum_{n=0}^{\infty} a_n x^n = a.$$