Proof of the Rogers - Ramanujan Identities

For $m = 0, 1, 2$ let

$$H_m = H_m(z,q) = \sum_{n=0}^{\infty} (-1)^n z^{2n} q^{\frac{1}{2}n(5n+1) - mn} (1 - z^m q^{2mn}) \cdot \frac{1}{(q)_n} \frac{1}{(zq^n;q)_\infty}$$

$$H_{m-1} = \sum_{n=0}^{\infty} (-1)^n z^{2n} q^{\frac{1}{2}n(5n+1) - (m-1)n} \frac{(1 - z^{m-1} q^{2(m-1)n})}{(q)_n (zq^n;q)_\infty}$$

$$H_m - H_{m-1} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} q^{\frac{1}{2}n(5n+1)}}{(q)_n (zq^n;q)_\infty} \cdot \left(q^{-mn} - z^m q^{mn} - q^{-(m-1)n} + z^{m-1} q^{(m-1)n} \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} q^{\frac{1}{2}n(5n+1)}}{(q)_n (zq^n;q)_\infty} \left(z^{m-1} q^{(m-1)n} (1 - zq^n) + q^{-mn} (1 - q^n) \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+m-1} q^{\frac{1}{2}n(5n+1) + n(m-1)}}{(q)_n (zq^{n+1};q)_\infty} + \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n} q^{\frac{1}{2}n(5n+1) - mn}}{(q)_{n-1} (zq^n;q)_\infty}$$