$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+m-1} q^{\frac{1}{2}n(5n+1)+n(m-1)}}{(q)_n \, (zq^{n+1};q)_\infty}$$

$$+ \sum_{n=0}^{\infty} (-1)^{n+1} \frac{z^{2(n+1)} q^{\frac{1}{2}(n+1)(5n+6)-m(n+1)}}{(q)_n \, (zq^{n+1};q)_\infty}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+m-1} q^{\frac{1}{2}n(5n+1)+n(m-1)} \left(1 - z^{3-m} q^{(2n+1)(3-m)}\right)}{(q)_n \, (zq^{n+1};q)_\infty}$$

since $\frac{1}{2}n(5n+1) + n(m-1) + (2n+1)(3-m)$

$= \frac{1}{2}n(5n+1) + 5n - nm - m + 3$

& $\frac{1}{2}(n+1)(5n+6) = \frac{1}{2}(n+1)((5n+1)+5)$

$= \frac{1}{2}n(5n+1) + \frac{1}{2}(5n + 5n+1 + 5)$

$= \frac{1}{2}n(5n+1) + (5n+3)$

Therefore,

$$H_{3-m}(zq,q) = \sum_{n=0}^{\infty} (-1)^n z^{2n} q^{\frac{1}{2}n(5n+1)-(3-m)n+2n} \frac{\cdot\left(1 - z^{3-m} q^{2(3-m)n+(3-m)}\right)}{(q)_n \, (zq^n;q)_\infty}$$

Hence

$\cancel{H_{3-m}(zq,q) = z}$

$$H_m(z,q) - H_{m-1}(z,q) = z^{m-1} H_{3-m}(zq,q).$$