$$H_0 = 0$$

$$H_1 - H_0 = H_2(zq)$$

$$H_2 - H_1 = z H_1(zq)$$

Hence

$$H_1(z) = H_2(zq)$$

$$H_2(z) = H_1(z) + z H_1(zq)$$

$$H_2(z) = H_2(zq) + z H_2(zq^2).$$

Let

$$H_2(z) = \sum_{n=0}^{\infty} c_n(q) z^n.$$

Therefore

$$\sum_{n=0}^{\infty} c_n(q) z^n = \sum_{n=0}^{\infty} c_n(q) q^n z^n + \sum_{n=0}^{\infty} c_n(q) q^{2n} z^{n+1}$$

$$\sum_{n=0}^{\infty} c_n(q) z^n = \sum_{n=0}^{\infty} c_n(q) q^n z^n + \sum_{n=1}^{\infty} c_{n-1} q^{2n-2} z^n$$

$$c_0(q) = H_2(0) = 1.$$

For $n \geq 1$,

$$c_n(q) = c_n(q) q^n + q^{2n-2} c_{n-1},$$

$$(1-q^n) c_n(q) = q^{2n-2} c_{n-1}(q)$$

$$c_n = \frac{q^{2n-2}}{1-q^n} c_{n-1}.$$

Iterating we obtain

$$c_n = \frac{q^{2n-2}}{1-q^n} \frac{q^{2n-4}}{1-q^{n-1}} \cdots \cdots \frac{q^0}{1-q^1} c_0$$

and

$$c_n = \frac{q^{n(n-1)}}{(q)_n}.$$

Hence,

$$H_2(z) = \sum_{n=0}^{\infty} \frac{z^n q^{n(n-1)}}{(q)_n}$$