(15)

$$\sum_{n=0}^{\infty} \frac{z^n q^{n(n-1)}}{(q)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} q^{\frac{1}{2}n(5n+1)-2n}(1-z^2q^{4n})}{(q)_n (zq^n;q)_\infty}$$

Letting $z=q$ we obtain

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(5n+1)}(1-q^{4n+2})}{(q)_n (q^{n+1};q)_\infty}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(5n+1)}(1-q^{4n+2})}{(q)_\infty}$$

$$= \frac{1}{(q)_\infty}\left(\sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(5n+1)} + \sum_{n=0}^{\infty} (-1)^{n+1} q^{\frac{n}{2}(5n+1)+4n+2} \right)$$

$m=-1-n$

(ie $n=-1-m$)

$$= \frac{1}{(q)_\infty}\left(\sum_{n=0}^{\infty} (-1)^n q^{\frac{n}{2}(5n+1)} + \sum_{m=-1}^{-\infty} (-1)^m q^{\frac{1}{2}(-1-m)(-5m-4)-4-4m+2} \right)$$

$$= \frac{1}{(q)_\infty}\left(\sum_{n=0}^{\infty} (-1)^n q^{\frac{n}{2}(5n+1)} + \sum_{m=-1}^{\infty} (-1)^m q^{\frac{1}{2}m(5m+1)} \right)$$

$$= \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n}{2}(5n+1)}$$

We need JTP

$$(z)_\infty (q/z)_\infty (q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} z^n$$

Replace q by q^5 and z by q^3:

$$(q^3;q^5)_\infty (q^2;q^5)_\infty (q^5;q^5)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{5n(n-1)/2} q^{3n}$$

$$\prod_{n=0}^{\infty} (1-q^{5n+2})(1-q^{5n+3})(1-q^{5(n+1)}) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(5n+1)/2}$$