Hence, $$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{\prod_{\substack{n\geq 1\\ n\equiv 0,2,3 \pmod 5}} (1-q^n)}{\prod_{n\geq 1} (1-q^n)}$$

$$= \prod_{\substack{n\geq 1\\ n\equiv 1,4 \pmod 5}} \frac{1}{(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}.$$

This completes the proof of the first Rogers Ramanujan identity. The 2nd Rogers Ramanujan identity

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$$

can be proved by using
$$H_1(z) = H_2(zq)$$
and letting $z=q$.

(1961) <u>Gordon's Generalization of the Rogers-Ramanujan Identities</u>

Let $k\geq 2$ and $i\geq 1$.

Let $B_{k,i}(n)$ = # of partitions of $n$ of the form $(b_1, b_2, \ldots, b_s)$ where $b_j - b_{j+k-1} \geq 2$ (for each $j$) and at most $i-1$ of the $b_j$ are equal to 1.

Let $A_{k,i}(n)$ = # of partitions of $n$ into parts $\not\equiv 0, \pm i \pmod{2k+1}$.

Then
$$A_{k,i}(n) = B_{k,i}(n) \quad \text{for all } n.$$