Roger's First Proof of the Rogers–Ramanujan Identities

Define $A_n(\theta)$ by

$$\sum_{n=0}^{\infty} \frac{A_n(\theta)\, z^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-2zq^n\cos\theta+z^2q^{2n})}$$

$$(|q|<1,\ |z|<1).$$

Then

$$(1-2z\cos\theta+z^2)\sum_{n=0}^{\infty} \frac{A_n(\theta)\, z^n}{(q;q)_n} = \sum_{n=0}^{\infty} \frac{A_n(\theta)\, q^n z^n}{(q)_n}$$

$$\sum_{n=0}^{\infty} \frac{A_n(\theta)\, z^n}{(q)_n} - 2\cos\theta \sum_{n=0}^{\infty} \frac{A_n(\theta)\, z^{n+1}}{(q)_n} + \sum_{n=0}^{\infty} \frac{A_n(\theta)\, z^{n+2}}{(q)_n} = \sum_{n=0}^{\infty} \frac{A_n(\theta)\, q^n z^n}{(q)_n}$$

$$\sum_{n=0}^{\infty} \frac{A_n}{(q)_n} z^n - 2\cos\theta \sum_{n=1}^{\infty} \frac{A_{n-1}}{(q)_{n-1}} z^n + \sum_{n=2}^{\infty} \frac{A_{n-2}}{(q)_{n-2}} z^n = \sum_{n=0}^{\infty} \frac{A_n(\theta)\, q^n z^n}{(q)_n}$$

$$A_n \frac{(1-q^n)}{(q)_n} = 2\cos\theta \frac{A_{n-1}}{(q)_{n-1}} - \frac{A_{n-2}}{(q)_{n-2}} \qquad \text{for } n \geq 2$$

$$\frac{A_n}{(q)_{n-1}} = 2\cos\theta \frac{A_{n-1}}{(q)_{n-1}} - \frac{A_{n-2}}{(q)_{n-2}}$$

Multiplying both sides by $(q)_{n-1}$ we obtain

$$A_n(\theta) = 2\cos\theta\, A_{n-1}(\theta) - (1-q^{n-1})\, A_{n-2}$$